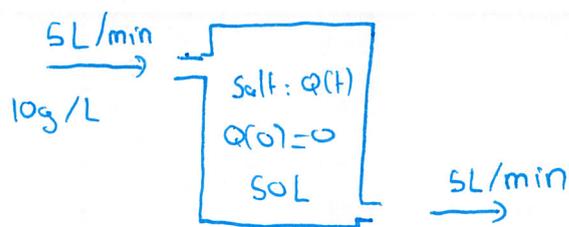


Quiz 1, D3



$$\frac{dQ}{dt} = (\text{rate of salt in}) - (\text{rate of salt out}) = 5 \times 10 - 5 \times \frac{Q(t)}{50}$$

$$\leadsto \begin{cases} \frac{dQ}{dt} = 50 - \frac{1}{10} Q \\ Q(0) = 0 \end{cases}$$

- This is a first order linear separable differential equation.
- When t goes to infinity, we can expect the concentration of salt in the tank to be 10g/L (the one coming in).

This means: $Q(t) \approx 10 \times 50 = 500\text{g}$.

(For a second method, see the solution sheet of the section D1-D2)

2) $t > 0$, $\begin{cases} y' + \frac{1}{t} y = \frac{\ln(t)}{t} \\ y(1) = 0 \end{cases}$

This is a first order linear differential equation (not separable)

Integrating factor: $\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

$$\leadsto (ty(t))' = \ln t$$

And thus: $ty(t) = \int \ln t dt$.

Integration by part: $\int \underbrace{1}_{u'} \cdot \underbrace{\ln t}_v dt = \underbrace{t \ln t}_{uv} - \int \underbrace{t}_{uv'} \times \underbrace{\frac{1}{t}}_{v'} dt = t \ln t - t + c$

Therefore: $y(t) = \ln t - 1 + \frac{c}{t}$

$$y(1) = 0 \Rightarrow 0 = 0 - 1 + c \quad : \quad c = 1$$

Finally:

$$y(t) = \ln t - 1 + \frac{1}{t}$$

$$3) \begin{cases} \frac{dy}{dt} = \cos t \cdot y(2-y) \\ y(2\pi) = 1 \end{cases}$$

This is a first order nonlinear separable ^{differential} equation.

Two constant solutions: $y=0$ and $y=2$

If $y \neq 0, 2$:

$$\frac{dy}{y(2-y)} = \cos t \, dt$$

$$\frac{1}{y(2-y)} = \frac{A}{y} + \frac{B}{2-y} = \frac{(B-A)y + 2A}{y(2-y)} \quad : \quad \frac{1}{y(2-y)} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{2-y} \right)$$

$\rightarrow A=B \quad \rightarrow A=1/2$

$$\sim \left(\frac{1}{y} + \frac{1}{2-y} \right) dy = 2 \cos t \, dt$$

By integration:

$$\ln|y| - \ln|2-y| = 2 \sin t + C$$

$$\ln \left| \frac{y}{2-y} \right| = 2 \sin t + C$$

$$\left| \frac{y}{2-y} \right| = c e^{2 \sin t}$$

But $y(2\pi) = 1 \in (0, 2)$: $\forall t, 0 < y(t) < 2$

Thus: $|y| = y$ and $|2-y| = 2-y$

$$\frac{y}{2-y} = c e^{2 \sin t} \Rightarrow y(t) = \frac{2 c e^{2 \sin t}}{1 + c e^{2 \sin t}}$$

But $y(2\pi) = 1$: $1 = \frac{2c}{1+c} : 1+c=2c : c=1$

Finally: $y(t) = \frac{2 e^{2 \sin t}}{1 + e^{2 \sin t}}$

and y is defined everywhere.