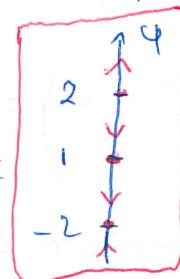
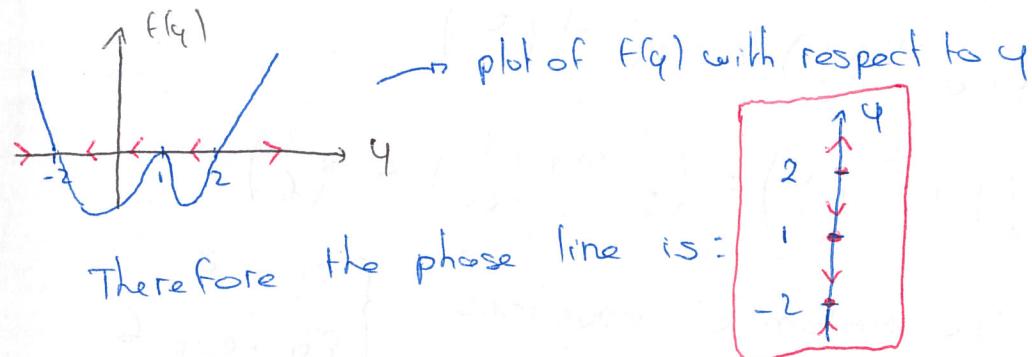


Quiz 2 - D1-D2

1) $y' = (y-1)^2(y^2-4)$

a) Equilibria: $y = -2, 1, 2$

b) $f(y) = (y-1)^2(y^2-4)$



c) According to the phase line:

- . $y = -2$ is a stable equilibrium
- . $y = 1$ is a semi-stable equilibrium
- . $y = 2$ is an unstable equilibrium

2) $\begin{cases} (t-1)y''(t) + (t+1)y'(t) + ty(t) = \cos t \\ y(0) = 2, \quad y'(0) = 2 \end{cases}$

a) $x_1 = y, \quad x_2 = y', \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{t-1} & -\frac{(t+1)}{t-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\cos t}{t-1} \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

b) The system is non autonomous (the matrix is not constant) and non homogeneous (the vector $\vec{g}(t)$ is not $\vec{0}$).

c) We have a system of the form $\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{g}(t)$

Here P and \vec{g} are continuous on $(-\infty, 1)$ or $(1, +\infty)$.

The initial date is chosen at time $t=0 \in (-\infty, 1)$.

Therefore we have an unique solution defined on $(-\infty, 1)$

- 3) $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, A is a 2×2 constant matrix and:
- $\lambda_1 = 3$ is an eigenvalue, with eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
 - $\lambda_2 = -2$ is an eigenvalue, with eigenvector $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

General form of the solutions:

$$\begin{aligned}\vec{x}(t) &= C_1 e^{3t} \vec{v}_1 + C_2 e^{-2t} \vec{v}_2 \\ &= C_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\end{aligned}$$

Here: $\vec{x}(0) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

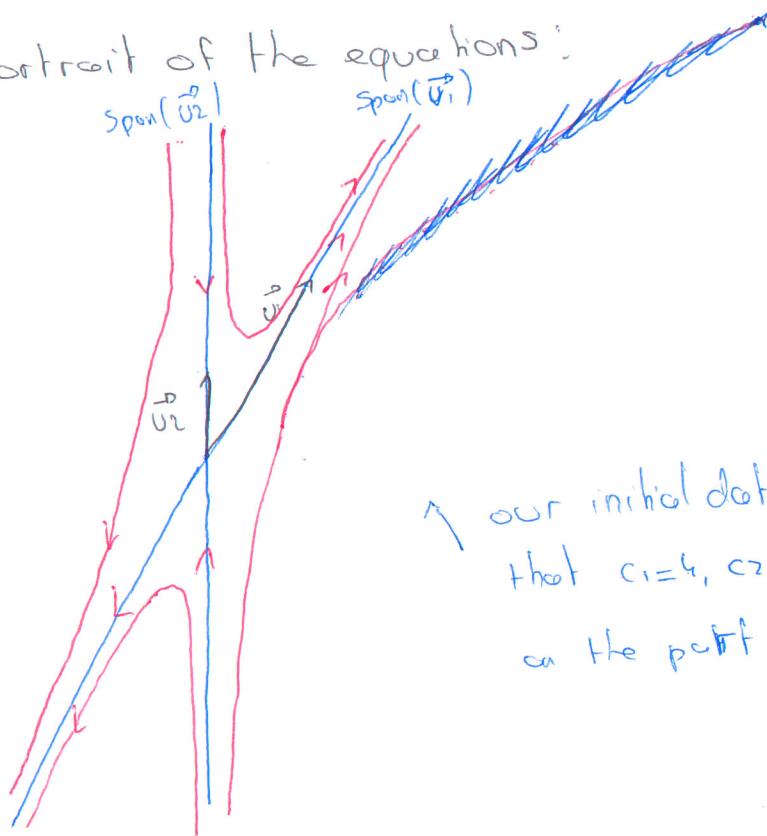
Therefore we have the system of equations:

$$\begin{cases} C_1 = 4 \\ 2C_1 + C_2 = 6 \end{cases}$$

And the solution is $C_1 = 4, C_2 = -2$

i.e.:
$$\boxed{\vec{x}(t) = 4e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Bonus: phase portrait of the equations:



This is a
saddle

our initial data tell us
that $C_1 = 4, C_2 = -2$, we are
on the part of the graph.

