

Quiz 2, Math 2552

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name Solution Last Name Q3, D1-D2

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Section Number (D1, D2 or D3) _____ TA Name _____

Student Instructions

- Show your work and justify your answers for all questions unless stated otherwise.
- Organize your work in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

1. (10 points) Solve the initial value problem $y'' - 2y' + 5y = 0$ with initial condition $y(0) = 1$ and $y'(0) = 5$ and draw the phase portrait (in the yy' -plane).

* Characteristic equation: $\lambda^2 - 2\lambda + 5 = 0$

$$\lambda_{1,2} = \frac{2}{2} \pm \frac{1}{2}\sqrt{4-20} = 1 \pm 2i$$

General form of the solution: $y(t) = e^{t \pm 2i} (C_1 \cos(2t) + C_2 \sin(2t))$.

* Solution of the IVP:

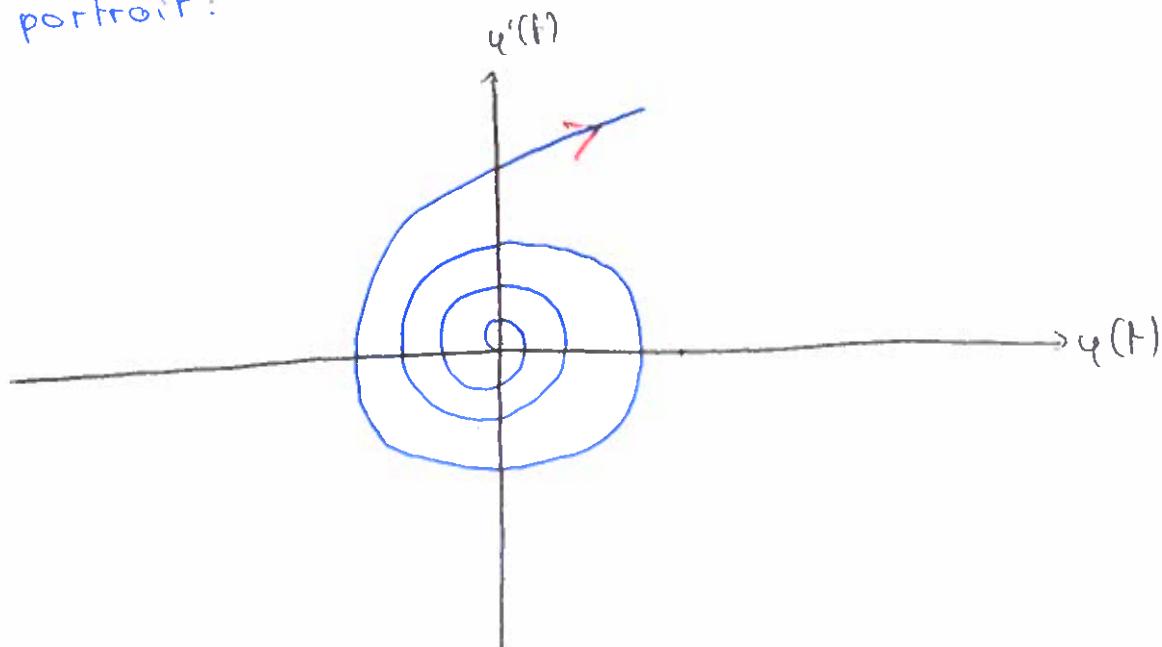
$$y(t) = e^t (C_1 \cos(2t) + C_2 \sin(2t)) : y(0) = C_1 = 1.$$

$$y'(t) = e^t ((C_1 + 2C_2) \cos(2t) + (C_2 - 2C_1) \sin(2t)) : y'(0) = C_1 + 2C_2 = 5 \\ \Rightarrow C_2 = 2.$$

The solution of the IVP is

$$y(t) = e^t (\cos(2t) + 2 \sin(2t))$$

* Phase portrait:



2. (10 points) Consider the system of equations $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} -2 & 1 & 7 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$. Find the general solution of this system. You can use that one eigenvalue of A is given by $\lambda = 1+i$, and a corresponding eigenvector is given by $\vec{v} = \begin{pmatrix} 11-2i \\ 5i \\ 5 \end{pmatrix}$.

* eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 & 7 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (-2-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (-2-\lambda)(1^2 - 2\lambda + 2)$$

$$= (-2-\lambda)(1 - (1+i))(1 - (1-i))$$

as eigenvalues: $\lambda_1 = -2, \lambda_2 = 1+i, \lambda_3 = 1-i$

* eigenvectors:

$$\lambda_1 = -2 : (A + 2I | 0) = \left(\begin{array}{ccc|c} 0 & 1 & 7 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 3 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 7 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -4 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

We can take $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$\lambda_2 = \underbrace{1+i}_{= 2+i\beta} : \vec{v} = \begin{pmatrix} 11-2i \\ 5i \\ 5 \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 5 \end{pmatrix} + i \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} = \vec{a} + i \vec{b}$$

* General form of the solutions:

$$\vec{x}(t) = C_1 e^{-2t} \vec{v}_1 + C_2 e^{it} \left(\cos(\beta t) \vec{a} - \sin(\beta t) \vec{b} \right) + C_3 e^{it} \left(\sin(\beta t) \vec{a} + \cos(\beta t) \vec{b} \right)$$

$$= C_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{it} \left(\cos t \begin{pmatrix} 11 \\ 0 \\ 5 \end{pmatrix} - \sin t \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} \right) + C_3 e^{it} \left(\sin t \begin{pmatrix} 11 \\ 0 \\ 5 \end{pmatrix} + \cos t \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} \right)$$

3. (10 points) Consider the 2nd order linear differential equation with initial value:

$$\begin{cases} (t^2 + a)y''(t) + \ln(t+10)y'(t) + \sqrt{10-t}y(t) = \cos(\sin(t)), \\ y(1) = 0, \quad y'(1) = 2, \end{cases}$$

where $a \neq -1$ is a real parameter.

(a) (2 points) Put this equation in the standard form $y'' + p(t)y' + q(t)y = g(t)$.

$$y'' + \frac{\ln(t+10)}{t^2+a} y' + \frac{\sqrt{10-t}}{t^2+a} y = \frac{\cos(\sin(t))}{t^2+a}$$

(b) (4 points) Express the forbidden values for t in the three following cases: if $a = -4$, if $a = 0$ and if $a = 1$.

- * If $a = -4$, the forbidden values for t are: $t \leq -10$, $t > 10$, $t = \pm 2$
- * If $a = 0$, the forbidden values for t are: $t \leq -10$, $t > 10$, $t = 0$
- * If $a = 1$, the forbidden values for t are: $\underbrace{t \leq -10}$, $\underbrace{t > 10}$, $\underbrace{t = \pm 2}$
 because of $\ln(t+10)$ because of $\sqrt{10-t}$ because of $\frac{1}{t^2+a}$

(c) (4 points) Write down the largest interval in which a unique solution of the IVP is guaranteed to exist in the three different cases: if $a = -4$, if $a = 0$ and if $a = 1$.

The initial value is given at time $t = 1$.

According to the previous question, the solution of the IVP
is defined on:

- * $(-2, 2)$ if $a = -4$
- * $(0, 10]$ if $a = 0$
- * $(-10, 10]$ if $a = 1$