Quiz 2, Math 2552

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name Solution Last Name Q3 - 10:10 (D3)

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Section Number (D1, D2 or D3) ______ TA Name ____________________

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

1. (10 points) Solve the initial value problem $y" - 4y' + 5y = 0$ with initial condition $y(0) = 1$ and $y'(0) = 4$ and draw the phase portrait (in the yy'−plane).

* Characteristic equation: $\lambda^2 - 4\lambda + 5 = 0$.

$$\lambda_{1,2} = \frac{4}{2} \pm \frac{1}{2}\sqrt{16-20} = 2 \pm i.$$

General solution: $y(t) = (C_1 \cos t + C_2 \sin t)\, e^{2t}$.

* solution of the IVP:

$y(t) = e^{2t}(C_1 \cos t + C_2 \sin t)$: $y(0) = C_1 = 1$

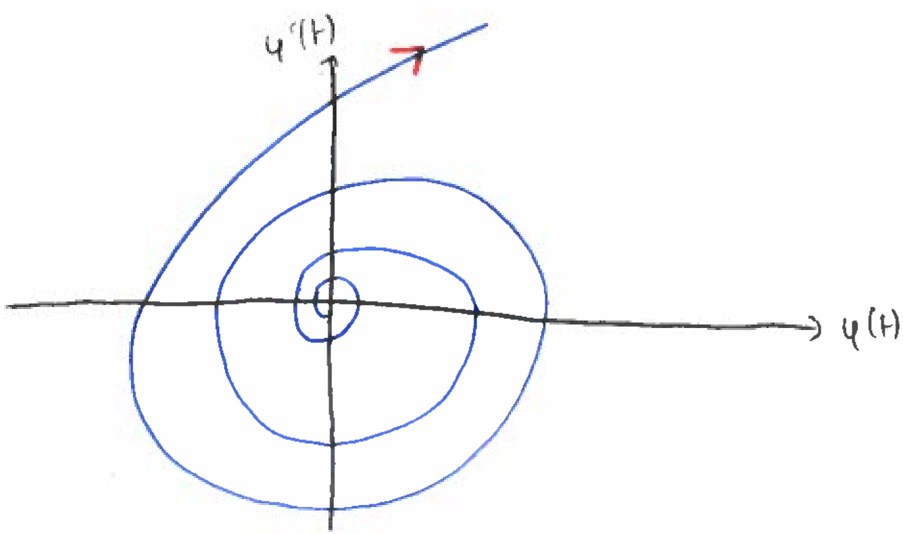
$y'(t) = e^{2t}\left((2C_1 + C_2)\cos t + (2C_2 - C_1)\sin t\right)$: $y'(0) = 2C_1 + C_2 = 4$

$\rightsquigarrow C_2 = 2$.

The solution of the IVP is:

$$\boxed{y(t) = e^{2t}(\cos t + 2\sin t)}$$

* Phase portrait:



2. (10 points) Consider the system of equations $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$. Find the general solution of this system. You can use that one eigenvalue of A is given by $\lambda = 2 + i$, and a corresponding eigenvector is given by $\vec{v} = \begin{pmatrix} 1-6i \\ i \\ 1 \end{pmatrix}$.

* eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & -1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 4\lambda + 5)$$

$$= (2-\lambda)(\lambda - (2+i))(\lambda - (2-i))$$

eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 2+i$ and $\lambda_3 = 2-i$

* eigenvectors:

$\lambda_1 = 2$: $(A - 2I \mid 0) = \left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

we can take $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$\lambda_2 = 2+i$ $\quad \vec{v_2} = \begin{pmatrix} 1-6i \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} = \vec{a} + i\vec{b}$

* General form of the solutions:

$$\vec{x}(t) = C_1 e^{2t} \vec{v_1} + C_2 e^{2t} (\cos t\, \vec{a} - \sin t\, \vec{b}) + C_3 e^{2t} (\sin t\, \vec{a} + \cos t\, \vec{b})$$

$$= C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t} \left(\cos t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} \right) + C_3 e^{2t} \left(\sin t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \cos t \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} \right)$$

3. (10 points) Consider the 2nd order linear differential equation with initial value:

$$\begin{cases} (t^2+a)y''(t) + \ln(t+11)y'(t) + \sqrt{(8-t)}y(t) = \sin(e^t), \\ y(1) = 0, \quad y'(1) = 2, \end{cases}$$

where $a \neq -1$ is a real parameter.

(a) (2 points) Put this equation in the standard form $y'' + p(t)y' + q(t)y = g(t)$.

$$y'' + \frac{\ln(t+11)}{t^2+a} y' + \frac{\sqrt{8-t}}{t^2+a} y = \frac{\sin(e^t)}{t^2+a}$$

(b) (4 points) Express the forbidden values for t in the three following cases: if $a = -9$, if $a = 0$ and if $a = 4$.

* If $a = -9$: the forbidden values are: $t \leq -11$, $t > 8$, $t = \pm 3$
* If $a = 0$: the forbidden values are: $t \leq -11$, $t > 8$, $t = 0$
* If $a = 4$: the forbidden values are: $\underbrace{t \leq -11}_{\text{because of } \ln(t+11)}$, $\underbrace{t > 8}_{\text{because of } \sqrt{8-t}}$. $\underbrace{\quad}_{\text{because of } \frac{1}{t^2+a}}$

(c) (4 points) Write down the largest interval in which a unique solution of the IVP is guaranteed to exist in the three different cases: if $a = -9$, if $a = 0$ and if $a = 4$.

The initial value is given at time 1.

with respect to the previous question, the solution is defined

on:

* $(-3, 3)$ if $a = -9$.
* $(0, 8]$ if $a = 0$
* $(-11, 8]$ if $a = 4$