

## Quiz 4, Math 2552

**PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS**

First Name Solution Last Name Q4 D1

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Section Number (D1, D2 or D3) \_\_\_\_\_ TA Name \_\_\_\_\_

### **Student Instructions**

- Show your work and justify your answers for all questions unless stated otherwise.
- Organize your work in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

1. (10 points) A mass of 1 kg stretches a spring 4.9 m. The mass is acted on by a time-dependent external force of  $10 \sin(2t)$  N, the damping factor is  $\gamma = 2$  N/m.

- (a) (2 points) Compute the spring constant  $k$  (using that  $g = 9.8$  N/kg).

$$mg = kL \Rightarrow k = \frac{m \cdot g}{L} = \frac{1 \cdot 9.8}{4.9} = 2.$$

- (b) (8 points) Using the method of undetermined coefficients, find a particular solution for this equation.

$$my'' + \gamma y' + ky = F(t) : y'' + 2y' + 2y = 10 \sin(2t)$$

$\sin(2t)$  is not a solution of the homogeneous equation  
(because we have a damping coefficient).

Thus we look for a particular solution of the form:

$$y_p(t) = A \cos(2t) + B \sin(2t) \quad *2$$

$$y_p'(t) = -2A \sin(2t) + 2B \cos(2t) \quad *2$$

$$y_p''(t) = -4A \cos(2t) - 4B \sin(2t) \quad *1$$

$$\begin{aligned} y_p'' + 2y_p' + 2y_p &= (-4A + 4B + 2A) \cos(2t) + (-4B - 4A + 2B) \sin(2t) \\ &= (\underbrace{4B - 2A}_{=0}) \cos(2t) + (\underbrace{-2B - 4A}_{=10}) \sin(2t) = 10 \sin(2t) \end{aligned}$$

$$\rightsquigarrow 4B - 2A = 0 : A = 2B$$

$$\rightsquigarrow -2B - 4A = 10 : -10B = 10 : B = -1, A = -2$$

$$\rightsquigarrow y_p(t) = -2 \cos(2t) - \sin(2t)$$

2. (10 points) Consider the 2nd order linear differential equation:

$$y''(t) - 3y'(t) + 2y(t) = \cos(t)e^{3t}, \quad (1)$$

(a) (2 points) Find 2 reals  $\lambda_1 < \lambda_2$  such that  $y_1(t) = e^{\lambda_1 t}$  and  $y_2(t) = e^{\lambda_2 t}$  define two solutions of the homogeneous equation.

characteristic equation: 
$$\underbrace{\lambda^2 - 3\lambda + 2}_{} = 0 \\ = (\lambda - 1)(\lambda - 2)$$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 2$

(b) (2 points) Compute the Wronskian  $W[y_1, y_2](t)$ .

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = e^t \times 2e^{2t} - e^{2t} e^t =$$

$\Rightarrow W[y_1, y_2](t) = e^{3t}$

(c) (2 points) Compute the anti-derivatives  $\int \frac{-y_2 g}{W} dt$  and  $\int \frac{y_1 g}{W} dt$ .

$$v_1(t) = \int \frac{-y_2 g}{W} dt = \int \frac{-e^{2t} \cos e^{3t}}{e^{3t}} dt = - \int \cos e^{2t} dt = \frac{1}{5} e^{2t} (-\sin t - 2\cos t) \quad (*)$$

$$v_2(t) = \int \frac{y_1 g}{W} dt = \int \frac{e^t \cos(t) e^{3t}}{e^{3t}} dt = \int \cos(t) e^t dt = \frac{1}{2} e^t (\cos t + \sin t) \quad (**)$$

(d) (4 points) Find a particular solution for the equation (1).

$\Rightarrow$  we have:  $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ .

$$\begin{aligned} \text{Thus: } y_p(t) &= \frac{1}{5} e^{2t} (-\sin t - 2\cos t) e^t + \frac{1}{2} e^t (\cos t + \sin t) e^{2t} \\ &= e^{3t} \left( -\frac{1}{5} \sin t - \frac{2}{5} \cos t + \frac{1}{2} \cos t + \frac{1}{2} \sin t \right) \end{aligned}$$

$\Rightarrow y_p(t) = e^{3t} \left( \frac{1}{10} \cos t + \frac{3}{10} \sin t \right)$

(\*) Computation for  $v_1, v_2$ : integration by parts (twice):

$$\begin{aligned} v_1(t) &= \int -\cos t e^{2t} dt = -\sin t e^{2t} + 2 \int \sin t e^{2t} dt \\ &= -\sin t e^{2t} - 2 \cos t e^{2t} - 4 \int \cos t e^{2t} dt \end{aligned}$$

$\Rightarrow v_1(t) = \frac{1}{5} (-\sin t - 2\cos t) e^{2t} \quad (\text{the same method works for } v_2)$ .

3. (10 points) Compute the Laplace transform of the function  $f$  defined by:

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 2, \\ 0 & \text{if } 2 < t < 4, \\ e^{3t} & \text{if } t \geq 4. \end{cases}$$

$$\begin{aligned} \mathcal{L}[f(t)](s) &= \int_0^{+\infty} f(t) e^{-st} dt \\ &= \int_0^2 t e^{-st} dt + \int_2^4 0 \cdot e^{-st} dt + \int_4^{+\infty} e^{3t} e^{-st} dt \end{aligned}$$

(20%)

\* By integration by parts:

$$\begin{aligned} \int_0^2 t e^{-st} dt &= -\frac{te^{-st}}{s} \Big|_0^2 + \frac{1}{s} \int_0^2 e^{-st} dt \\ &= -\frac{2e^{-2s}}{s} + \frac{1}{s} \times -\frac{1}{s} e^{-st} \Big|_0^2 \\ &= -\frac{2}{s} e^{-2s} - \frac{1}{s^2} (e^{-2s} - 1) \end{aligned}$$

$$\times \int_4^{+\infty} e^{3t} e^{-st} dt = \int_4^{+\infty} e^{-(s-3)t} dt = -\frac{1}{s-3} e^{-(s-3)t} \Big|_4^{+\infty} = \frac{e^{-4(s-3)}}{s-3} \quad \text{for } s > 3$$

Finally:

$$\mathcal{L}[f(t)](s) = -\frac{2e^{-2s}}{s} - \frac{e^{-2s} - 1}{s^2} + \frac{e^{-4(s-3)}}{s-3} \quad \text{for } s > 3$$