

## Quiz 4, Math 2552

**PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS**

First Name Solution Last Name Q4 - D2

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### Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

1. (10 points) A mass of 1 kg stretches a spring 4.9 m. The mass is acted on by a time-dependent external force of  $10 \sin(2t)$  N, the damping factor is  $\gamma = 2$  N/m.

(a) (2 points) Compute the spring constant  $k$  (using that  $g = 9.8$  N/kg).

$$mg = kL \Rightarrow k = \frac{m \cdot g}{L} = \frac{1 \cdot 9.8}{4.9} = 2.$$

(b) (8 points) Using the method of undetermined coefficients, find a particular solution for this equation.

$$my'' + \gamma y' + ky = F(t) : y'' + 2y' + 2y = 10 \sin(2t)$$

$\sin(2t)$  is not a solution of the homogeneous equation (because we have a damping coefficient).

Thus we look for a particular solution of the form:

$$y_p(t) = A \cos(2t) + B \sin(2t) \quad \times 2$$

$$y_p'(t) = -2A \sin(2t) + 2B \cos(2t) \quad \times 2$$

$$y_p''(t) = -4A \cos(2t) - 4B \sin(2t) \quad \times 1$$

$$\begin{aligned} y_p'' + 2y_p' + 2y_p &= (-4A + 4B + 2A) \cos(2t) + (-4B - 4A + 2B) \sin(2t) \\ &= \underbrace{(4B - 2A)}_{=0} \cos(2t) + \underbrace{(-2B - 4A)}_{=10} \sin(2t) = 10 \sin(2t) \end{aligned}$$

$$\leadsto 4B - 2A = 0 : A = 2B$$

$$\leadsto -2B - 4A = 10 : -10B = 10 : B = -1, A = -2$$

$$\leadsto y_p(t) = -2 \cos(2t) - \sin(2t)$$

2. (10 points) Consider the 2nd order linear differential equation:

$$y''(t) - 3y'(t) + 2y(t) = te^t, \quad (1)$$

- (a) (2 points) Find 2 reals  $\lambda_1 < \lambda_2$  such that  $y_1(t) = e^{\lambda_1 t}$  and  $y_2(t) = e^{\lambda_2 t}$  define two solutions of the homogeneous equation.

characteristic equation:  $\lambda^2 - 3\lambda + 2 = 0$   
 $= (\lambda - 1)(\lambda - 2)$

$\leadsto \lambda_1 = 1, \lambda_2 = 2$

- (b) (2 points) Compute the Wronskian  $W[y_1, y_2](t)$ .

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^t \times 2e^{2t} - e^t e^{2t}$$

$\leadsto W[y_1, y_2](t) = e^{3t}$

- (c) (2 points) Compute the anti-derivatives  $\int \frac{-y_2 g}{W} dt$  and  $\int \frac{y_1 g}{W} dt$ .

$$v_1(t) = \int \frac{-y_2 g}{W} dt = \int \frac{-e^{2t} te^t}{e^{3t}} dt = \int -t dt = -\frac{1}{2}t^2$$

$$v_2(t) = \int \frac{y_1 g}{W} dt = \int \frac{e^t te^t}{e^{3t}} dt = \int te^{-t} dt \stackrel{\text{IBP}}{=} -te^{-t} + \int e^{-t} dt = e^{-t}(-t-1)$$

- (d) (4 points) Find a particular solution for the equation (1).

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

$$y_p(t) = -\frac{1}{2}t^2 e^t + e^{-t}(-t-1)e^{2t}$$

$$y_p(t) = e^t \left( -\frac{1}{2}t^2 - t - 1 \right)$$

3. (10 points) Compute the Laplace transform of the function  $f$  defined by:

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 2, \\ 0 & \text{if } 2 < t < 4, \\ e^{3t} & \text{if } t \geq 4. \end{cases}$$

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \int_0^{+\infty} f(t) e^{-st} dt \\ &= \int_0^2 t e^{-st} dt + \int_2^4 0 \cdot e^{-st} dt + \int_4^{+\infty} e^{3t} e^{-st} dt \end{aligned}$$

By integration by parts:

$$\begin{aligned} \int_0^2 t e^{-st} dt &= -\frac{t e^{-st}}{s} \Big|_0^2 + \frac{1}{s} \int_0^2 e^{-st} dt \\ &= -\frac{2 e^{-2s}}{s} + \frac{1}{s} \times -\frac{1}{s} e^{-st} \Big|_0^2 \\ &= -\frac{2}{s} e^{-2s} - \frac{1}{s^2} (e^{-2s} - 1) \end{aligned}$$

$$\int_4^{+\infty} e^{3t} e^{-st} dt = \int_4^{+\infty} e^{-(s-3)t} dt = -\frac{1}{s-3} e^{-(s-3)t} \Big|_4^{+\infty} = \frac{e^{-4(s-3)}}{s-3} \text{ for } s > 3$$

Finally:

$$\mathcal{L}\{f(t)\}(s) = \frac{-2e^{-2s}}{s} - \frac{e^{-2s} - 1}{s^2} + \frac{e^{-4(s-3)}}{s-3} \text{ for } s > 3$$