Quiz 4, Math 2552

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name Solution Last Name D3

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Section Number (D1, D2 or D3) ______ TA Name ______________________

Student Instructions

- Show your work and justify your answers for all questions unless stated otherwise.
- Organize your work in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing; your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.
1. (10 points) A mass of 2 kg stretches a spring 19.6 m. The mass is acted on by a time-dependent external force of $5 \sin(t)$ N, the damping factor is $\gamma = 2$ N/m.

(a) (2 points) Compute the spring constant $k$ (using that $g = 9.8$ N/kg).

$$mg = kL \Rightarrow k = \frac{mg}{L} = \frac{2 \times 9.8}{19.6} = 1$$

(b) (8 points) Using the method of undetermined coefficients, find a particular solution for this equation.

$$my'' + \gamma y' + ky = F(t) : \quad 2y'' + 2y' + y = 5 \sin(t).$$

$\sin(t)$ is not a solution of the homogeneous equation (because the equation is damped).

Thus, we look for a solution of the form:

$$y_p(t) = A \cos(t) + B \sin(t)$$

$$y_p'(t) = -A \sin(t) + B \cos(t)$$

$$y_p''(t) = -A \cos(t) - B \sin(t)$$

$$2y'' + 2y' + y_p = (-2A + 2B + A) \cos(t) + (-2B - 2A + B) \sin(t)$$

$$= (2B - A) \cos(t) + (-B - 2A) \sin(t) = 5 \sin(t)$$

$$\Rightarrow 2B - A = 0 \quad \Rightarrow A = 2B$$

$$\Rightarrow -B - 2A = 5 \quad \Rightarrow -5B = 5 \quad \Rightarrow B = -1, \ A = -2$$

Therefore:

$$y_p(t) = -2 \cos(t) - \sin(t)$$
2. (10 points) Consider the 2nd order linear differential equation:

\[ y''(t) - 5y'(t) + 6y(t) = te^{2t}, \quad (1) \]

(a) (2 points) Find 2 reals \( \lambda_1 < \lambda_2 \) such that \( y_1(t) = e^{\lambda_1 t} \) and \( y_2(t) = e^{\lambda_2 t} \) define two solutions of the homogeneous equation.

\[
\text{Characteristic equation: } \quad \lambda^2 - 5\lambda + 6 = 0
\]
\[
\implies \lambda_1 = 2, \quad \lambda_2 = 3
\]

(b) (2 points) Compute the Wronskian \( W[y_1, y_2](t) \)

\[
W(t) = W[y_1, y_2](t) = (y_1 y_2')(t) - (y_1'y_2)(t) = e^{2t} \times 3e^{3t} - 2e^{2t} \times e^{3t} = e^{5t}
\]

(c) (2 points) Compute the anti-derivatives \( \int \frac{-y_2g}{W} \, dt \) and \( \int \frac{y_1g}{W} \, dt \).

\[
v_1(t) = \int \frac{-y_1g}{W} \, dt = \int \frac{-e^{3t} \times t \times e^{2t}}{e^{5t}} \, dt = \int -t \, dt = -\frac{1}{2} t^2
\]
\[
v_2(t) = \int \frac{y_1g}{W} \, dt = \int \frac{e^{2t} \times t \times e^{t}}{e^{5t}} \, dt = \int e^{-t} \, dt = -e^{-t} + \int e^{-t} \, dt = e^{-t} (-t - 1)
\]

(d) (4 points) Find a particular solution for the equation (1).

\[
y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)
\]
\[
= -\frac{1}{2} t^2 e^{2t} + e^{-t} (-t - 1) e^{3t}
\]
\[
\implies y_p(t) = e^{2t} (-\frac{1}{2} t^2 - t - 1)
\]
3. (10 points) Compute the Laplace transform of the function $f$ defined by:

$$f(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq 3, \\ 0 & \text{if } 3 < t < 5, \\ e^{-t} & \text{if } t \geq 5. \end{cases}$$

$$F(s) = \int_0^{+\infty} f(t) e^{-st} dt$$

$$= \int_0^3 2te^{-st} dt + \int_3^5 0e^{-st} dt + \int_5^{+\infty} e^{-t} e^{-st} dt$$

* For the first part, by integration by parts:

$$\int_0^3 te^{-st} dt = \left[ \frac{tx - e^{-st}}{s} \right]_0^3 + \frac{1}{s} \int_0^3 e^{-st} dt$$

$$= \frac{-3e^{-3s}}{s} + \frac{1}{s} \left[ e^{-st} \right]_0^3$$

$$= \frac{-3e^{-3s}}{s} - \frac{e^{-3s} - 1}{s}$$

* For the third part:

$$\int_5^{+\infty} e^{-t} e^{-st} dt = \int_5^{+\infty} e^{-(s+1)t} dt = \frac{-1}{s+1} e^{-(s+1)t} \bigg|_5^{+\infty}$$

$$= \frac{-1}{s+1} e^{-5(s+1)}$$

* Finally:

$$F(s) = \frac{6e^{-3s}}{s} - \frac{2(e^{-3s} - 1)}{s^2} + \frac{e^{-5(s+1)}}{s+1} \quad \text{for } s > 0$$