Quiz 4, Math 2552

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name Solution Last Name D3

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Section Number (D1, D2 or D3) _____ TA Name _____

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

1. (10 points) A mass of 2 kg stretches a spring 19.6 m. The mass is acted on by a time-dependent external force of $5\sin(t)$ N, the damping factor is $\gamma = 2$ N/m.

(a) (2 points) Compute the spring constant k (using that $g = 9.8$ N/kg).

$$mg = kL \Rightarrow \boxed{k = \frac{m.g}{L} = \frac{2 \times 9.8}{19.6} = 1}$$

(b) (8 points) Using the method of undetermined coefficients, find a particular solution for this equation.

$$my'' + \gamma y' + ky = F(t) \quad : \quad 2y'' + 2y' + y = 5\sin(t).$$

$\sin(t)$ is not a solution of the homogeneous equation (because the equation is damped).

Thus, we look for a solution of the form:

$$y_p(t) = A\cos(t) + B\sin(t)$$

$$y_p'(t) = -A\sin(t) + B\cos(t)$$

$$y_p''(t) = -A\cos t - B\sin(t)$$

$$2y_p'' + 2y_p' + y_p = (-2A + 2B + A)\cos(t) + (-2B - 2A + B)\sin(t)$$

$$= \underbrace{(2B - A)}_{0}\cos(t) + \underbrace{(-B - 2A)}_{5}\sin(t) = 5\sin(t)$$

$\rightsquigarrow$ $2B - A = 0$: $A = 2B$

$\rightsquigarrow$ $-B - 2A = 5$: $-5B = 5$; $B = -1$, $A = -2$

Therefore:

$$\boxed{y_p(t) = -2\cos(t) - \sin(t)}$$

2. (10 points) Consider the 2nd order linear differential equation:

$$y''(t) - 5y'(t) + 6y(t) = te^{2t}, \tag{1}$$

(a) (2 points) Find 2 reals $\lambda_1 < \lambda_2$ such that $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ define two solutions of the homogeneous equation.

Characteristic equation: $\underbrace{\lambda^2 - 5\lambda + 6}_{= (\lambda-2)(\lambda-3)} = 0$

$\rightsquigarrow \boxed{\lambda_1 = 2, \quad \lambda_2 = 3}$

(b) (2 points) Compute the Wronskian $W[y_1, y_2](t)$.

$W(t) = W[y_1, y_2](t) = (y_1 y_2')(t) - (y_2' y_1)(t) = e^{2t} \times 3e^{3t} - 2e^{2t} e^{3t}$

$\rightsquigarrow \boxed{W(t) = e^{5t}}$

(c) (2 points) Compute the anti-derivatives $\int \frac{-y_2 g}{W} dt$ and $\int \frac{y_1 g}{W} dt$.

$v_1(t) = \int \frac{-y_2 g}{W} dt = \int \frac{-e^{3t} \times t e^{2t}}{e^{5t}} dt = \int -t\,dt = -\frac{1}{2} t^2$

$v_2(t) = \int \frac{y_1 g}{W} dt = \int \frac{e^{2t} \times t e^{2t}}{e^{5t}} dt = \int \underset{u}{t} \underset{v'}{e^{-t}} dt \overset{IBP}{=} -te^{-t} + \int e^{-t} dt$

$= e^{-t}(-t-1)$.

(d) (4 points) Find a particular solution for the equation (1).

$y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)$

$= -\frac{1}{2} t^2 e^{2t} + e^{-t}(-t-1) e^{3t}$

$\rightsquigarrow \boxed{y_p(t) = e^{2t}\left(-\frac{1}{2} t^2 - t - 1\right)}$

3. (10 points) Compute the Laplace transform of the function f defined by:

$$f(t) = \begin{cases} 2t & \text{if} \quad 0 \le t \le 3, \\ 0 & \text{if} \quad 3 < t < 5, \\ e^{-t} & \text{if} \quad t \ge 5. \end{cases}$$

$$F(s) = \int_0^{+\infty} f(t) e^{-st} dt$$

$$= \int_0^3 2te^{-st} dt + \cancel{\int_3^5 0 e^{-st} dt}^{\,0} + \int_5^{+\infty} e^{-t} e^{-st} dt$$

* For the first part, by integration by parts:

$$\int_0^3 \overset{u}{t} \overset{v'}{e^{-st}} dt = \left[t \times \frac{-e^{-st}}{s} \right]_0^3 + \frac{1}{s} \int_0^3 e^{-st} dt$$

$$= \frac{-3e^{-3s}}{s} + \frac{1}{s} \times \frac{-1}{s} e^{-st} \Big|_0^3$$

$$= \frac{-3e^{-3s}}{s} - \frac{e^{-3s} - 1}{s^2}$$

* For the third part:

$$\int_5^{+\infty} e^{-t} e^{-st} dt = \int_5^{+\infty} e^{-(s+1)t} dt = \frac{-1}{s+1} e^{-(s+1)t} \Big|_5^{+\infty}$$

$$= \frac{+1}{s+1} e^{-5(s+1)}$$

* Finally:

$$\boxed{F(s) = \frac{-6e^{-3s}}{s} - \frac{2(e^{-3s} - 1)}{s^2} + \frac{e^{-5(s+1)}}{s+1} \quad \text{for } s > 0}$$