Quiz 1 consists of 3 problems.
Do not use any books and notes.
Do not use calculator, computer, cell phone, or any electronic device.
Please print your name and your GT ID number on this page. Please print your name on all pages. In doing that, you are agreeing to abide by the terms of the Georgia Tech Honor Code.
The grading will be done on the scanned images of your test. Please write clearly and darkly.
Please write your solution of each problem on the same page where the problem is given. Do not write your solutions on the back sides of the pages.
[1] (20 points) Find the general solutions of \( \frac{dy}{dt} = y^2 t \sin t \).

**Solution:** This is a separable equation.

\[
\frac{1}{y^2} \, dy = t \sin t \, dt,
\]
\[
\int \frac{1}{y^2} \, dy = \int t \sin t \, dt,
\]
\[
-\frac{1}{y} = uv - \int v \, du,
\]
\[
= t(- \cos t) - \int (- \cos t) \, dt,
\]
\[
= -t \cos t + \sin t + C.
\]

Solve for \( y \):

\[
y = \frac{1}{-t \cos t + \sin t + C}
\]

Alternative answers:

\[
y = \frac{1}{t \cos t - \sin t - C}
\]
\[
y = \frac{1}{t \cos t - \sin t + C}
\]
[2] (20 points) Solve the initial value problem \( \frac{dy}{dt} + 3y = t^3e^{-3t} + 6e^{-t}, \quad y(0) = 1. \)

**Solution:** This is a first order linear diff eq, solved by the integrating factor. Prepare \( A(t) \), an antiderivative of \( a(t) = 3 \): \( A(t) = 3t \).

The diff eq is equivalent to

\[
\frac{d}{dt} [e^{3t} y] = e^{3t} (t^3e^{-3t} + 6e^{-t}) = t^3 + 6e^{2t}.
\]

Integrate:

\[
e^{3t} y = \frac{1}{4} t^4 + 3e^{2t} + C.
\]

Examine the initial condition \( y(0) = 1 \):

\[
1 = 3 + C, \quad C = -2.
\]

Thus,

\[
e^{3t} y = \frac{1}{4} t^4 + 3e^{2t} - 2.
\]

**Answer:** \( y = \frac{1}{4} t^4 e^{-3t} + 3e^{-t} - 2e^{-3t} \)
A tank initially contains 100 L of pure water. A mixture containing a concentration of 5 g/L of salt enters the tank at a rate of 10 L/min, and the well-stirred mixture leaves the tank at the same rate. Denote by $Q(t)$ the amount of salt in the tank at $t$ min.

Write the differential equation and the initial condition for $Q(t)$.

**Remark:** You are only asked to set up equations satisfied by $Q(t)$. You are not required to find $Q(t)$.

**Solution:**

- **Rate of brine in:** 10 L/min
- **Salt concentration:** 5 g/L
- **Brine Volume:** 100 L
- **Salt:** $Q(t)$ g
- **Rate of brine out:** 10 L/min
- **Initial Condition:** $Q(0) = 0$ g

Differential equation:

$$\frac{dQ}{dt} = \text{(rate of salt flowing in)} - \text{(rate of salt flowing out)}$$

$$= (10)(5) - 10 \frac{Q(t)}{100}$$

$$= 50 - \frac{1}{10}Q(t).$$

Initial Condition: $Q(0) = 0$.

*Answer:* $\frac{dQ}{dt} = 50 - \frac{1}{10}Q, \; Q(0) = 0.$