Section 2.4 : Differences Between Linear and Nonlinear Equations

Chapter 2 : First Order Differential Equations

Math 2552 Differential Equations
Section 2.4

**Topics**
We will cover these topics in this section.

1. Theorems for first order linear and nonlinear IVPs.

**Objectives**
For the topics covered in this section, students are expected to be able to do the following.

1) Characterize first order linear and nonlinear IVPs in terms of existence and uniqueness.

2) Determine intervals where a solution to a given first order IVP exists.
Motivation

There are two questions that we exploring in this section.

Given an initial value problem (IVP).

1. **existence**: does the IVP have a solution, and if so, where?
2. **uniqueness**: is the solution unique?

We explore these questions for linear and non-linear cases.
A Motivating Example

Consider the IVP
\[
\frac{dy}{dt} = 8ty^{1/5}, \quad y(0) = 0
\]

Take a few minutes to answer the following on your own.

a) Is the ODE linear?

b) Solve the IVP to determine an expression for \( y(t) \).

c) Is your solution unique? Can you identify another solution to the IVP?

After a few minutes compare your results with someone sitting nearby.
Linear IVP

**Theorem: Existence and Uniqueness of 1st Order Linear IVP**

If $p$ and $g$ are continuous on $(\alpha, \beta)$, $t_0 \in (\alpha, \beta)$, then there is a **unique** solution to the IVP

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

A proof of this theorem is in the textbook.

**Example:** Consider the IVP

$$(9 - t^2)y' + 5ty = 3t^2, \quad y(-1) = 1$$

Determine an interval where a solution to the IVP exists.
Nonlinear IVP

Theorem: Existence and Uniqueness of 1st Order Nonlinear IVP

If \( f \) and \( \frac{\partial f}{\partial y} \) are continuous over \( \alpha < t < \beta \), and \( \gamma < y < \delta \) which contains the point \((t_0, y_0)\), then there is a unique solution to the IVP

\[
y' = f(t, y), \quad y(t_0) = y_0
\]

on an interval contained in \( \alpha < t < \beta \).

Note that:

- A proof of this theorem goes beyond the scope of this course.
- These conditions are sufficient, but not necessary.
- The expression \( \frac{\partial f}{\partial y} \) is a partial derivative.
- Does \( \frac{dy}{dt} = 8ty^{1/5}, \quad y(0) = 0 \), satisfy the conditions of this theorem?
The 1st order ODE \( y' + p(t)y = g(t) \) has the following properties.

1. If the \( p \) and \( g \) are continuous, there is a general solution containing an arbitrary constant that represents all solutions of the ODE.
2. There is an explicit expression for the ODE.
3. Points where the solution is discontinuous can be found by identified without solving the ODE: they are identified from the coefficients.

A **nonlinear** first order ODE does not necessarily have any of the above properties.