Section 2.7 : Substitution Methods

Chapter 2 : First Order Differential Equations

Math 2552 Differential Equations
Section 2.7

Topics
We will cover these topics in this section.

1. Substitution methods: homogeneous differential equations and Bernoulli equations

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1) Identify whether a given DE is homogeneous, or Bernoulli
2) Solve homogeneous and Bernoulli equations
A function \( f(x, y) \) is **homogeneous function of degree** \( k \) if

\[
f(\lambda x, \lambda y) = \lambda^k f(x, y)
\]

**Definition: Homogeneous DE**

A differential equation

\[
M(x, y) + N(x, y) \frac{dy}{dx} = 0
\]

is **homogeneous** if and only if \( M \) and \( N \) are homogeneous functions of the same order.

**Example:** determine whether the differential equation is homogeneous.

\[
(x^2 + y^2)dx + (x^2 - xy)dy = 0
\]
Solving a Homogeneous DE

If $M$ and $N$ are homogeneous degree $k$ then

$$M(\lambda x, \lambda y) = \lambda^k M(x, y), \quad N(\lambda x, \lambda y) = \lambda^k N(x, y),$$

We can also write this as:
Example 1

Solve the differential equation.

\[(x^2 + y^2)dx + (x^2 - xy)dy = 0\]
A differential equation

\[ \frac{dy}{dx} + q(x)y = r(x)y^n, \quad n \in \mathbb{R} \]

is a Bernoulli equation.

Observe that

- If \( n = 1 \), the DE is separable, linear, and homogeneous.
- The substitution \( u = y^{1-n} \) reduces a Bernoulli equation to a linear equation.
Example 2

Solve the DE.

\[ x \frac{dy}{dx} + y = x^2 y^2 \]
<table>
<thead>
<tr>
<th>Type</th>
<th>Form of DE</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>homogeneous</td>
<td>$M(x, y) + N(x, y)\frac{dy}{dx} = 0$</td>
<td>let $y = ux$ or let $x = vy$</td>
</tr>
<tr>
<td></td>
<td>$M, N$ homog. same order</td>
<td></td>
</tr>
<tr>
<td>Bernoulli</td>
<td>$\frac{dy}{dx} + q(x)y = r(x)y^n$</td>
<td>let $u = y^{1-n}$</td>
</tr>
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