Section 4.5 : Undetermined Coefficients

Chapter 4 : Second Order Equations

Math 2552 Differential Equations

"I have not failed. I’ve just found 10,000 ways that won’t work."

- Thomas Edison

Applying the method of undetermined coefficients can involve trying different solutions and testing them until we find one that works.
Section 4.5

**Topics**
We will cover these topics in this section.

1. The method of undetermined coefficients

**Objectives**
For the topics covered in this section, students are expected to be able to do the following.

1. Solve differential equations and initial value problems using the method of undetermined coefficients
Theorem

In this section, we seek solutions to the nonhomogenous problem

\[ y'' + p(t)y' + q(t)y = g(t) \]  \hspace{1cm} (1)

The corresponding homogeneous problem is

\[ y'' + p(t)y' + q(t)y = 0 \]  \hspace{1cm} (2)

Note that if \( Y_1 \) and \( Y_2 \) are solutions of (1), then their difference is:

This implies that the general solution of (1) is:
To solve a nonhomogeneous DE:

1. Construct solution to homogeneous problem
2. Determine a particular solution to the nonhomogeneous problem using either
   ▶ method of undetermined coefficients
   ▶ variation of parameters
3. Add the functions found in the two previous steps to form the general solution
Participation Activity: Worksheet

Depending on how much time we have, we will run examples in this lecture as a participation activity.

- Please work in groups of ________________
- Each group submits one sheet of paper
- Instructor has paper you can use
- Print full names and email addresses on cover
- Every student in a group gets the same grade
- Grading scheme per question:
  - 0 marks for no work or working alone
  - 1 mark for starting the problem or for a final answer with insufficient justification
  - 2 marks for a complete solution
Examples

Determine a particular solution of the following.

1. \( y'' + 3y' + 2y = 10e^{3t} \)
2. \( y'' + 3y' + 2y = \sin t \)
3. \( y'' - 6y' + 9y = e^{3t} \)
4. \( y'' + 4y = 5t^2 e^t \) (if time permits)
A Solution Strategy for Undetermined Coefficients

To solve an IVP containing an equation of the form

\[ ay'' + by' + cy = g(t) \]

we can use the following steps.

1. Obtain general solution of homogeneous equation
2. Determine if undetermined coefficients can be used
3. If \( g(t) = \sum_{i}^{n} g_i \), then consider each of the \( n \) sub-problems separately
4. Solve first sub-problem: assume particular solution form, determine coefficients
5. Repeat previous step for each sub-problem
6. Form general solution to differential equation
7. Solve IVP
In the table below, $s$ is the smallest non-negative integer so that $Y$ is a solution of the homogeneous equation, and

$$P_n(t) = a_0 t^n + a_1 t^{n-1} \ldots + a_n$$  \hspace{1cm} (3)

$$Q_n(t) = A_0 t^n + A_1 t^{n-1} \ldots + A_n$$  \hspace{1cm} (4)

$$R_n(t) = B_0 t^n + B_1 t^{n-1} \ldots + B_n$$  \hspace{1cm} (5)

<table>
<thead>
<tr>
<th>$g(t)$</th>
<th>particular solution $Y(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n(t)$</td>
<td>$t^s Q_n$</td>
</tr>
<tr>
<td>$P_n(t) e^{\alpha t}$</td>
<td>$t^s e^{\alpha t} Q_n$</td>
</tr>
<tr>
<td>$P_n(t) e^{\alpha t} \sin(\beta t)$</td>
<td>$t^s e^{\alpha t} (\cos(\beta t)Q_n + \sin(\beta t)R_n)$</td>
</tr>
<tr>
<td>$P_n(t) e^{\alpha t} \cos(\beta t)$</td>
<td>$t^s e^{\alpha t} (\cos(\beta t)Q_n + \sin(\beta t)R_n)$</td>
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