Section 4.7 : Variation of Parameters

Chapter 4 : Second Order Equations

Math 2552 Differential Equations
Sections 4.7

Topics
We will cover these topics in this section.

1. Variation of parameters for 2nd order differential equations
2. Variation of parameters for first order systems
3. Fundamental matrix

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Solve non-homogeneous first order systems and second order differential equations using the method of variation of parameters.
Motivation

Undetermined coefficients:

- does not give us an explicit expression for the particular solution,
- can only be applied when $g(t)$ is a combination of sine, cosine, exponential, and polynomials.

Variation of parameters addresses these points.
We seek solutions to the **nonhomogenous** problem

$$y'' + p(t)y' + q(t)y = g(t)$$  \hspace{1cm} (1)

The solution to the corresponding **homogeneous** problem is

$$y_h = c_1 y_1(t) + c_2 y_2(t)$$  \hspace{1cm} (2)

To construct a particular solution, we replace $c_1$ and $c_2$ with functions:

$$y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$$  \hspace{1cm} (3)

Our goal is to determine functions $v_1$ and $v_2$. 
Procedure for Variation of Parameters

To obtain a particular solution of (1) using variation of parameters:

1. Construct solution to homogeneous problem to obtain $y_1, y_2$
2. Solve the system of nonlinear equations:

$$y_1 v'_1 + y_2 v'_2 = 0$$
$$y'_1 v_1 + y'_2 v_2 = g$$

3. Integrate $v'_1$ and $v'_2$ to obtain $v_1$ and $v_2$.

4. Set $y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$

Note:

- The textbook has an explicit formula for $v_1$ and $v_2$ that you don’t have to memorize, but you can if you prefer.
- If time permits, we will derive the above procedure in lecture.
Example

Determine a particular solution to

\[ t^2 y'' - 4ty' + 6y = 4t^3, \quad t > 0 \]

given that \( y_1 = t^2 \) and \( y_2 = t^3 \) are solutions to the homogeneous equation.
Nonhomogeneous Systems

We seek solutions to the linear nonhomogenous system

\[ \ddot{x}' = P \dot{x} + \ddot{g}(t) \]  

(6)

\[ P = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}, \quad \ddot{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \]  

(7)

The solution to the corresponding homogeneous problem is

\[ \ddot{x}_h = c_1 \ddot{x}_1(t) + c_2 \ddot{x}_2(t) \]  

(8)

To obtain a particular solution, replace \( c_1 \) and \( c_2 \) with functions:

\[ \ddot{x}_p = v_1(t) \ddot{x}_1(t) + v_2(t) \ddot{x}_2(t) \]  

(9)

Goal: determine scalar functions \( v_1(t) \) and \( v_2(t) \).
Suppose the solutions to the homogeneous problem are:

\[ \vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} \]

Then introduce the **fundamental matrix**, \( X(t) \),

\[ X(t) = [\vec{x}_1 \ \vec{x}_2] = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \]  \hspace{1cm} (10)

*If time permits, we will use this matrix to re-write our differential equation and derive formulas for the particular solution during lecture.*
Particular Solution Formula

Theorem

Assume entries of matrix $P$ and $g$ are continuous on open interval $I$, $X$ is the fundamental matrix, then

$$\vec{x}' = P\vec{x} + \vec{g}(t)$$

has a particular solution

$$\vec{x}_p = X(t) \int X^{-1}(t)\vec{g}(t)\,dt$$
Example

Determine a particular solution to

\[ \vec{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix} \]

given that the solutions to the homogeneous equation are

\[ \vec{x}_1 = e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]