Section 5.5 : Discontinuous and Periodic Functions

Chapter 5 : The Laplace Transform

Math 2552 Differential Equations
Section 5.5

Topics
We will cover these topics in this section.

1. Step and Indicator functions

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Represent functions using the step function.
2. Solve IVPs, with piecewise continuous functions, using the Laplace Transform.
Suppose that at time $t = 0$ we place a pie into an oven whose temperature, $y(t)$, is $20^\circ$C.

- When $t = 0$, the pie and the oven are both $20^\circ$C.
- $y(t)$ increases linearly at rate $50^\circ$C per min, until $t = 4$.
- For $t \geq 4$ min, $y = 220$.

Construct an IVP that represents this situation. How can we solve it?
The **unit step function** is defined as:

\[ u_c(t) = \begin{cases} 
0 & 0 \leq t < c \\
1 & c \leq t 
\end{cases} \]

The **indicator function** is defined as:

\[ u_{bc}(t) = \begin{cases} 
0 & 0 \leq t < b \\
1 & b \leq t \leq c \\
0 & c \leq t 
\end{cases} \]

**Question**
How can we express the indicator function in terms of step functions?
Express the following functions in terms of step functions.

\[ f(t) = \begin{cases} 
2 & 0 \leq t < 3 \\
-2 & 3 \leq t
\end{cases} \]

\[ g(t) = \begin{cases} 
t & 0 \leq t < 2 \\
t^2 & 2 \leq t < 4 \\
t^3 & 4 \leq t
\end{cases} \]
Transform of a Step Function

Theorem

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$$

Proof:
Example

Solve the IVPs.

\[ y' + y = f, \quad y(0) = 0, \quad f = \begin{cases} 1 & 0 \leq t < 1 \\ -1 & 1 \leq t \end{cases} \]
Shift in the $t$-Domain

Theorem

$$\mathcal{L}\{u_c f(t - c)\} = e^{-cs} F(s)$$

Proof:
Recall that $\mathcal{L}\{u_c f(t - c)\} = e^{-cs} F(s)$.

Compute the following.

1. $\mathcal{L}^{-1}\left\{ \frac{1}{s - 4} e^{-2s} \right\}$

2. $\mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 9} e^{-\pi s/2} \right\}$
Suppose \( f(t) \) is periodic with period \( T \). Then

\[
\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) dt
\]

Proof:
Examples

Recall that $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) dt$.

Compute the Laplace Transform of the following periodic functions.

- $y_1(t)$

- $y_2(t)$
Evaluate $\mathcal{L}\{\cos(t) u_\pi\}$. 