Section 7.1 : Autonomous Systems and Stability

Chapter 7 : Systems of First Order Linear Equations

Math 2552 Differential Equations
Topics
We will cover these topics in this section.

1. Critical points of non-linear two dimensional autonomous systems.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Identify and classify critical points of non-linear systems of autonomous differential equations.
Recall: Autonomous Systems

In this section we take a closer look at non-linear autonomous two-dimensional systems:

\[
\frac{dy}{dt} = f(x, y), \quad \frac{dx}{dt} = g(x, y)
\]

For example,

\[
\frac{dx}{dt} = 4 - 2y, \quad \frac{dy}{dt} = y^2 - xy
\]

We explored such systems in Section 3.6.
Recall: Critical Point Classification

Critical points of a system correspond to points where

\[
\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0
\]

If a trajectory \( \vec{x}(t) \) that starts \textbf{sufficiently close} to a critical point, \( \vec{x}_0 \), then the critical point is:

1. stable if:

2. asymptotically stable if:

3. unstable if:
A mass is attached to an inflexible rod that pivots about a point.

Under a gravitational force, the angle $\theta(t)$ the rod makes with the vertical axis satisfies

$$\frac{d^2 \theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin \theta = 0$$

Sketch the phase portrait, $\theta'$ vs. $\theta$, for a few different initial conditions. Where are the critical points located?
Example

Determine the critical points of the system:

\[
\frac{dx}{dt} = 4 - 2y, \quad \frac{dy}{dt} = 12 - 3x^2
\]

A phase portrait of the system is shown below.
You will need to plot direction fields to complete the homework.

A Google search will point to sites that can plot slope fields.

Here are a few that I’ve found are ok:

- www.bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html
- www.directionfield.com

So far, I’ve found wolframalpha to be most useful:

```
streamplot[{4 - 2y, 12 - 3x^2}, {x, -4, 4}, {y, -1, 5}]
```

On an exam I would give you the slope field to interpret.
Additional Examples (as time permits)

For each system, a) identify all of the critical points, and b) classify the critical points.

1. \( \frac{dx}{dt} = 2x - x^2 - xy, \quad \frac{dy}{dt} = 3y - 2y^2 - 3xy \)

2. \( \frac{dx}{dt} = x(6 - x - y), \quad \frac{dy}{dt} = -x + 7y - 2xy \)