

1) A pizza of 450°F is placed in a room of 70°F .
The cooling rate of the pizza is $k = 0.3 \text{ min}^{-1}$.

a) Newton's law of cooling: $\frac{dT}{dt} = -k(T - T_{\text{room}})$

Therefore here we have: $T' = -k(T - T_{\text{room}}) = -0.3(T - 70)$.

b) Initial value problem: $\begin{cases} T' = -0.3(T - 70) \\ T(0) = 450 \end{cases}$

c) This is a first order linear separable differential equation.

For example, using the separability:

$$\frac{dT}{T-70} = -0.3 dt \quad \xrightarrow{\text{integrate}} \quad \ln|T-70| = -0.3t + C$$

$$\text{Therefore: } |T-70| = C e^{-0.3t}$$

$T(0) > 70$ and $T=70$ is a constant solution: $\forall t, T(t) > 70$
(because two solution may not cross)

$$\text{Thus: } |T-70| = T-70 = C \cdot e^{-0.3t}$$

$$\Rightarrow T(t) = 70 + C e^{-0.3t}$$

$$\text{Then, } T(0) = \underbrace{70 + C}_{C=380} = 450 \Rightarrow \boxed{T(t) = 70 + 380 e^{-0.3t}}$$

d) We look for t such that $T(t) = 110$

$$\text{i.e.: } 70 + 380 e^{-0.3t} = 110$$

$$e^{-0.3t} = \frac{(110-70)}{380} = \frac{40}{380} = \frac{4}{38} = \frac{2}{19}$$

$$\approx -0.3t = \ln\left(\frac{2}{19}\right) = -\ln\left(\frac{19}{2}\right)$$

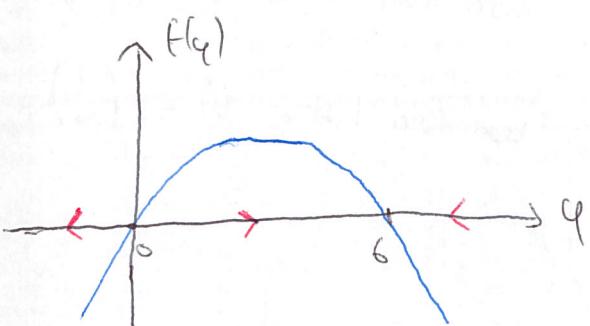
$$\text{Finally: } \boxed{t = \frac{1}{0.3} \ln\left(\frac{19}{2}\right) \approx 7.5 \text{ min}}$$

$$2) \dot{q} = 0.2 \left(1 - \frac{q}{6}\right) q$$

a) equilibria = constant solutions: i.e. $\dot{q} = 0$

$$\dot{q} = 0.2 \left(1 - \frac{q}{6}\right) q = 0 \Rightarrow \boxed{q=0 \text{ or } q=6}$$

b) let us first draw the graph of $f(q) = 0.2 \left(1 - \frac{q}{6}\right) q$ with respect to q



Phase portrait:



c) According to the phase portrait / phase line:

- $q=0$ is an unstable equilibrium.

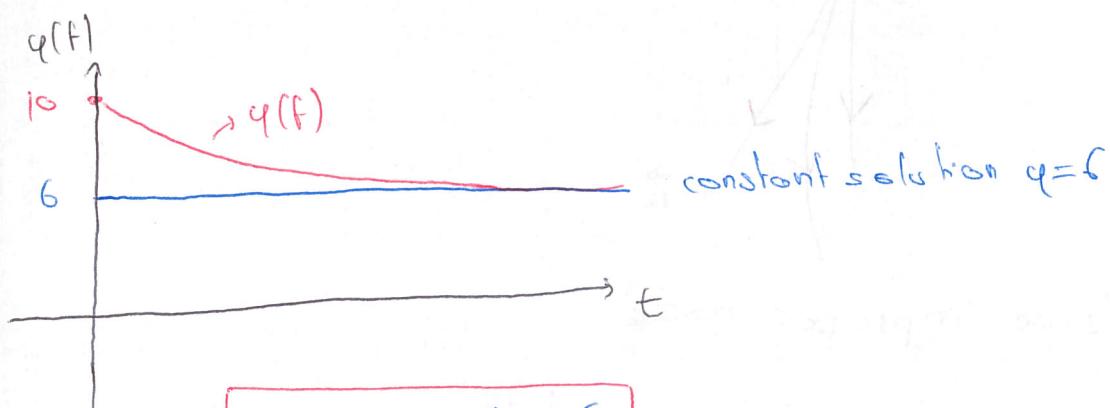
- $q=6$ is a stable equilibrium. (asymptotically stable)

$$d) q(0)=10$$

According to the phase line, the solution is decreasing.

Indeed, if $q(0)=10 > 6$, then: $\forall t, q(t) > 6$

and $\dot{q} = \frac{0.2 \left(1 - \frac{q}{6}\right) q}{\cancel{q} < 0} < 0$: the solution is decreasing



if $q(0)=10$,

$$\boxed{\lim_{t \rightarrow \infty} q(t) = 6}$$

$$3) \quad a) \quad \vec{x}' = \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} = A\vec{x}.$$

Eigenvalues: A is triangular, we can read the eigenvalues on the diagonal: $\lambda_1 = 1, \lambda_2 = 4$

We already know that:

- $\vec{0}$ is the only equilibrium (because A is invertible)
- we will obtain a repulsive improper node ($\lambda_1 > 0, \lambda_2 > 0$)

* Eigenvectors:

$$\lambda_1 = 1 : (A - I) = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} : 3x_1 + x_2 = 0 : \vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

$$\lambda_2 = 4 : (A - 4I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : x_2 = 0 : \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

+ General form of the solutions:

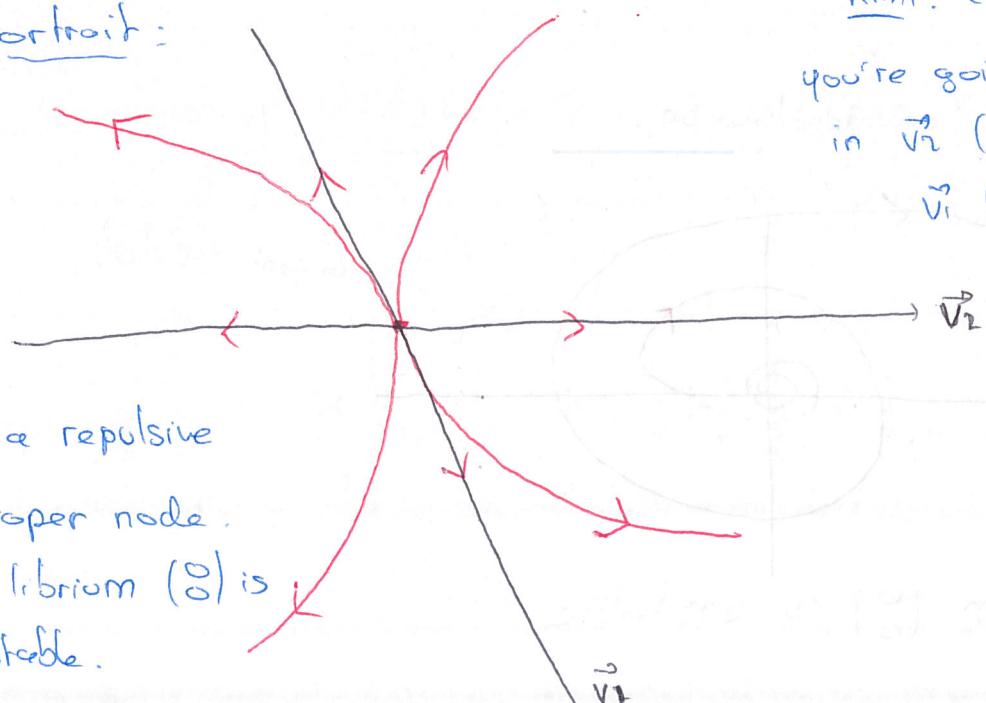
$$\begin{aligned} \vec{x}(t) &= C_1 e^{1t} \vec{v}_1 + C_2 e^{4t} \vec{v}_2 \\ &= C_1 e^t \begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

where C_1 and C_2 are free parameters.

Remark: concavity of the curve

you're going to infinity faster in $\vec{v}_2 (e^{4t})$ than in $\vec{v}_1 (e^t)$.

+ Phase portrait:



This is a repulsive
improper node.

The equilibrium (0) is
unstable.

$$b) \vec{x}' = \begin{pmatrix} 5 & -15 \\ 3 & -1 \end{pmatrix} \vec{x} = A\vec{x}$$

eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -15 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 40$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-160}}{2} = 2 \pm \frac{1}{2}\sqrt{-144} = 2 \pm \frac{1}{2}i\sqrt{4 \times 4 \times 9} = 2 \pm 6i$$

We already know that:

- $\vec{0}$ is the only equilibrium (because 0 is not an eigenvalue of A)
- we will obtain a repulsive spiral focus

eigenvector for $\lambda_1 = 2+6i$

$$(A - \lambda_1 I) = \begin{pmatrix} 3-6i & -15 \\ 3 & -3-6i \end{pmatrix} \sim \begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix} \sim \begin{pmatrix} 1 & -1-2i \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

General form of the solutions:

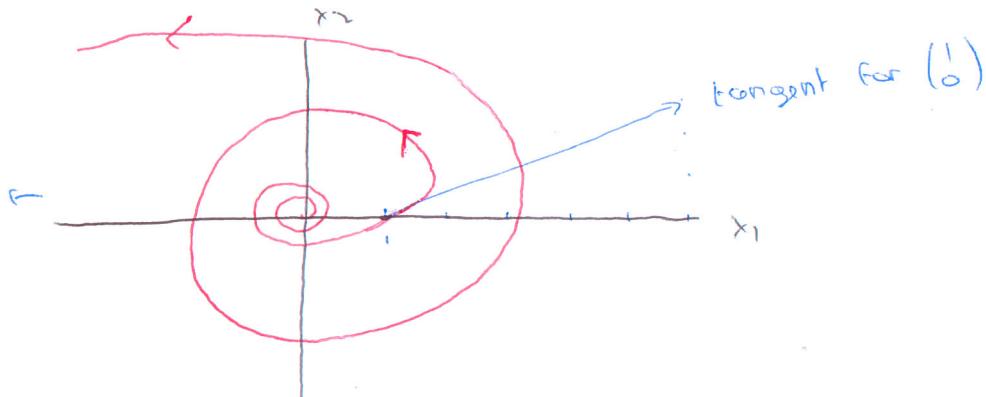
$$\vec{x}(t) = C_1 e^{2t} \left(\cos(\beta t) \vec{a} - \sin(\beta t) \vec{b} \right) + C_2 e^{2t} \left(\sin(\beta t) \vec{a} + \cos(\beta t) \vec{b} \right)$$

$$= C_1 e^{2t} \left(\cos(\beta t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin(\beta t) \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right) + C_2 e^{2t} \left(\sin(\beta t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cos(\beta t) \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right),$$

where C_1, C_2 are free parameters.

Phase portrait:

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}: \text{counterclockwise}$$



The equilibrium (0) is unstable