Second Order Linear Equations (4.1, 4.2, 4.3)

1. Determine the longest interval in which the given IVP is certain to have a unique, twice differentiable solution:

   a. \( t(t - 4)y'' + 3ty' + 4y = 2, \quad y(3) = 0, \quad y'(3) = -1 \)

   \textit{Solution.} Put into the standard form, and find the interval where all the coefficient functions are continuous and also contains \( t_0 \). \( I = (0, 4) \).

   b. \( (\ln t)y'' + \frac{t}{t^2 - 4}y' + y = 0, \quad y(1.5) = 4, \quad y'(1.5) = 1 \)

   \textit{Solution.} Put into the standard form, and find the interval where all the coefficient functions are continuous and also contains \( t_0 \). \( I = (1, 2) \)

   c. \( y'' + t^2y' + \tan(t)y = 2, \quad y(0) = 1, \quad y'(0) = -1 \)

   \textit{Solution.} Put into the standard form, and find the interval where all the coefficient functions are continuous and also contains \( t_0 \). \( I = (-\frac{\pi}{2}, \frac{\pi}{2}) \)

2. Solve the IVP, sketch the solution in the \( ty \)-plane and sketch the phase portrait (solution in the \( y'y \)-plane).

   a. \( y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2 \)

   \textit{Solution.} \( \lambda_1 = -1 + i, \quad \lambda_2 = -1 - i. \)

   \[
   y(t) = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t. \\
y'(t) = -C_1 e^{-t} (\cos t + \sin t) + C_2 e^{-t} (\cos t - \sin t).
   \]

   The solution to the IVP is

   \[
   y(t) = \sqrt{2} e^{-\frac{t}{2}} \cos t + \sqrt{2} e^{-\frac{t}{2}} \sin t. 
   \]

   b. \( y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3 \)

   \textit{Solution.} \( \lambda_1 = 0, \quad \lambda_2 = 3. \)

   \[
   y(t) = C_1 + C_2 e^{-3t}. \\
y'(t) = -3C_2 e^{-3t}. 
   \]

   The solution to the IVP is

   \[
   y(t) = -1 - e^{-3t}. 
   \]

   c. \( y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0 \)

   \textit{Solution.} \( \lambda_1 = -9, \quad \lambda_2 = 1. \)

   \[
   y(t) = C_1 e^{-9t} + C_2 e^t. \\
y'(t) = -9C_1 e^{-9t} + C_2 e^t. 
   \]

   The solution to the IVP is

   \[
   y(t) = \frac{1}{10} e^{9(1-t)} + \frac{9}{10} e^t. 
   \]