Method of Undetermined Coefficients and Forced Vibrations (4.5, 4.6)

1. Find the general solution by the method of undetermined coefficients.

a. \(4y'' - 4y' + y = 16e^{t/2}\)

Solution. \(\lambda_1 = \lambda_2 = \frac{1}{2}\).

Guess \(Y(t) = At^2 e^{t/2}\).

Plug in the guess function and solve for \(A\), we have \(A = 2\).

The general solution is \(y(t) = C_1 e^{t/2} + C_2 e^{t/2} + 2t e^{t/2}\).

b. \(y'' + 2y' = 3 + 4 \sin 2t\)

Solution. \(\lambda_1 = -2, \lambda_2 = 0\).

Guess \(Y(t) = At + B \sin 2t + C \cos 2t\).

Plug in the guess function and solve for \(A, B, C\), we have \(A = \frac{3}{2}, B = C = -\frac{1}{2}\).

The general solution is \(y(t) = C_1 e^{-2t} + C_2 + \frac{3}{2}t - \frac{1}{2} \sin 2t - \frac{1}{2} \cos 2t\).

c. \(y'' - 2y' - 3y = -3te^{-t}\)

Solution. \(\lambda_1 = -1, \lambda_2 = 3\).

Guess \(Y(t) = t(At + B)e^{-t}\).

Plug in the guess function and solve for \(A, B\), we have \(A = \frac{3}{8}, B = \frac{3}{16}\).

The general solution is \(y(t) = C_1 e^{-t} + C_2 e^{3t} + \frac{3}{8} t^2 e^{-t} + \frac{3}{16} te^{-t}\).

2. A 4 kg mass stretches the spring 8 cm at the equilibrium. The damper imparts a viscous force of 6 N when the speed of the mass is 5 cm/s. The position \(x\) of the mass is measured from the equilibrium.

a. Find the spring constant \(k\) and damping coefficient \(\gamma\). \((g = 10 m/s^2)\)

b. A time-dependant external force \(p(t) = P \sin(\omega t)\) is exerted on the mass, where \(P\) is 13 kN and \(\omega\) is 25 rad/s. Find the steady state solution.

Solution. \(m = 4, k = \frac{40}{0.08} = 500, \gamma = \frac{6}{0.05} = 120\).

\(4y'' + 120y' + 500y = 13000 \sin 25t\).

Solve for \(4\lambda^2 + 120\lambda + 500 = 0\), and we have \(\lambda_1 = -25, \lambda_2 = -5\). The guess function is \(A \sin 25t + B \cos 25t\), and \(A = -2, B = -3\).

Therefore, the steady state solution is \(y_p(t) = -2 \sin 25t - 3 \cos 25t\).