Method of Undetermined Coefficients and Forced Vibrations (4.5, 4.6)

1. Find the general solution by the method of undetermined coefficients.

a. $4y'' - 4y' + y = 16e^{t/2}$
Solution. $\lambda_1 = \lambda_2 = \frac{1}{2}$.
Guess $Y(t) = At^2e^{\frac{1}{2}}$.
Plug in the guess function and solve for A, we have $A = 2$.
The general solution is

$$y(t) = C_1e^{\frac{t}{2}} + C_2e^{\frac{t}{2}} + 2e^{\frac{t}{2}}t^2.$$

b. $y'' + 2y' = 3 + 4\sin 2t$
Solution. $\lambda_1 = -2$, $\lambda_2 = 0$.
Guess $Y(t) = At + B\sin 2t + C\cos 2t$.
Plug in the guess function and solve for A, B, C, we have $A = \frac{3}{2}$, $B = C = -\frac{1}{2}$.
The general solution is

$$y(t) = C_1e^{-2t} + C_2 + \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t.$$

c. $y'' - 2y' - 3y = -3te^{-t}$
Solution. $\lambda_1 = -1$, $\lambda_2 = 3$.
Guess $Y(t) = t(At + B)e^{-t}$.
Plug in the guess function and solve for A, B, we have $A = \frac{3}{8}$, $B = \frac{3}{16}$.
The general solution is

$$y(t) = C_1e^{-t} + C_2e^{3t} + \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t}.$$

2. A 4 kg mass stretches the spring 8 cm at the equilibrium. The damper imparts a viscous force of 6 N when the speed of the mass is 5 cm/s. The position x of the mass is measured from the equilibrium.

a. Find the spring constant k and damping coefficient γ. ($g = 10m/s^2$)

b. A time-dependant external force $p(t) = P\sin(\omega t)$ is exerted on the mass, where P is 13 kN and ω is 25 rad/s. Find the steady state solution.

Solution. $m = 4$, $k = \frac{40}{0.08} = 500$, $\gamma = \frac{6}{0.05} = 120$.

$$4y'' + 120y' + 500y = 13000\sin 25t.$$

Solve for $4\lambda^2 + 120\lambda + 500 = 0$, and we have $\lambda_1 = -25$, $\lambda_2 = -5$. The guess function is $A\sin 25t + B\cos 25t$, and $A = -2$, $B = -3$.
Therefore, the steady state solution is

$$y_p(t) = -2\sin 25t - 3\cos 25t.$$