Exam 2: Practice problems

1. Consider the simple harmonic oscillator $\dot{x} = v, \dot{v} = -\omega^2 x$. Show that the orbits are given by the ellipses $\omega^2 x^2 + v^2 = C$, where $C \neq 0$ is an arbitrary constant.

2. Consider the $2 \times 2$ linear system

$$X' = \begin{pmatrix} 1 & -4 \\ 2 & 3 \end{pmatrix} X.$$

(a) Compute the matrix exponential $e^{tA}$.
(b) Solve the ODE two ways.
(c) Sketch the phase plane.
(d) Classify the equilibrium point $(0,0)$.
(e) Compute the stable and unstable manifolds of $(0,0)$.

3. $X' = A \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} X, \quad \text{with } X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

(a) Compute the matrix exponential $e^{tA}$.
(b) Find the general solution two ways.
(c) Sketch the phase plane.
(d) Solve the IVP.
(e) Classify the equilibrium point $(0,0)$.
(f) Compute the stable and unstable manifolds of $(0,0)$.

4. Assume Romeo and Juliet are romantic clones and both retreat from their own feelings. Describe the outcome of this relationship?

5. For $2 \times 2$ matrices, state and prove the Cayley-Hamilton theorem.

6. For $2 \times 2$ matrices, prove the JCF for matrices with complex eigenvalues.

7. If $A^2 = A$, then find a simple formula for $e^A$. 