their natural order. Thus we have complete independence. The equivalence of these axioms and those of [1] can be shown as follows.

Proof of Theorem P-2. Let $a$ be the first element of the nonempty subset $N$ of $N$.

Proof of Theorem P-4. Let $G$ be a nonempty subset of $N$ such that for every $x \in N$, $I(x) \subset G$ implies $x \in G$. Suppose $G \neq N$. Then $N-G$ is nonempty and, by B-1, has a first element $b$. Then $I(b) \subset G$ and by hypothesis $b \in G$, a contradiction.

Proof of Theorem B-1. By P-2, $N$ is nonempty. Let $W$ be a nonempty subset of $N$ and suppose that $W$ has no first element. Then for each $x \in W$, there exists $y \in W$ such that $yRx$. Now $N-W$ is not empty for otherwise $W=N$ and $W$ would have a first element by P-2. Let $x \in N$ such that $I(x) \subset N-W$. Then $x \in N-W$, for if $x \in W$, then there exists some $y \in W$ such that $yRx$ and hence $y \in N-W$. By P-4, $N-W=N$ and $W$ is empty, a contradiction. Hence $W$ has a first element.

References

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A SHORT PROOF OF CRAMER’S RULE

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Many texts on linear algebra (e.g., [1] through [7]) prove Cramer’s rule by using the relationship $A^{-1} = \text{adj } A/\det A$ and comparing cofactor expansions. The following proof may provide more insight into what is actually happening when Cramer’s rule is used.

Let $Ax = b$, with $A$ $n \times n$ and nonsingular. Let the columns of $A$ be $a_1, \ldots, a_n$ and those of the identity be $e_1, \ldots, e_n$. Define $X_k$ by

$$X_k = [e_1, \ldots, e_{k-1}, x, e_{k+1}, \ldots, e_n].$$

Then

$$x_k = \det X_k = \det A^{-1}AX_k = \det AX_k/\det A$$

$$= \det [a_1, \ldots, a_{k-1}, b, a_{k+1}, \ldots, a_n]/\det A,$$

which is Cramer’s rule.

This proof makes it easier to see what we are doing when we use Cramer’s rule. We want to evaluate $\det X_k$ in order to find $x_k$. But $X_k$ contains the unobservable vector $x$. We therefore take the determinant of the image of $X_k$ under the transformation represented by $A$, and then, to compensate for the transformation, divide the result by $\det A$.
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References


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ON AN INTERESTING METRIC SPACE

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In [1] Hildebrand and Milnes have defined on the euclidean plane the metric:

\[ p_2(x, y) = \begin{cases} 
0 & \text{if } x_1 = y_1 \text{ and } x_2 = y_2 \\
\frac{1}{2} & \text{if } x_1 = y_1 \text{ and } x_2 \neq y_2 \text{ or } x_1 \neq y_1 \text{ and } x_2 = y_2 \\
1 & \text{if } x_1 \neq y_1 \text{ and } x_2 \neq y_2 
\end{cases} \tag{1} \]

where \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) are members of \( \mathbb{R}^2 \). The purpose of this note is twofold. First, to generalize this metric on \( \mathbb{R}^2 \) and extend the definition to \( \mathbb{R}^n \); this is done in Sections 1 and 2. Second, to introduce a metric which is rotation invariant but not translation invariant. This is done in Section 3.

1. Let \( a \) and \( b \) be positive real numbers such that \( b \leq 2a \leq 2b \). Define \( p_2^*: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) as \( p_2^*(x, y) = 0, a, \) or \( b \) respectively according as exactly both, exactly one or exactly none of the coordinate of \( x \) and \( y \) are equal. Then the following result is easily proved.

**Theorem 1.** For each fixed value of \( a \) and \( b \) as specified above \( p_2^* \) is a metric on \( \mathbb{R}^2 \).

We remark that for \( a = \frac{1}{2} \) and \( b = 1 \) we obtain the metric in (1). The metric \( p_2^* \) gives rise to the following interesting neighborhoods of a point in \( \mathbb{R}^2 \). An \( \epsilon \)-neighborhood of \( x \in \mathbb{R}^2 \) is the singleton set \( \{x\} \) if \( 0 < \epsilon \leq a \), the pair of straight lines thru \( x \) parallel to the coordinate axes if \( a < \epsilon \leq b \) and the whole plane if \( \epsilon > b \).

2. We now extend the definition of \( p_2 \) to the \( n \)-dimensional case.

**Theorem 2.** Let \( x, y \in \mathbb{R}^n \), \( n \geq 2 \). Define \( p_n^*(x, y) = 1 - (k/n) \) if exactly \( k \) of the coordinates of \( x \) are equal to the corresponding coordinates of \( y \) \((k = 0, 1, \ldots, n)\). Then \( p_n \) is a metric on \( \mathbb{R}^n \).