1 Saddle-Node, Transcritical, Pitchfork Bifurcations

Here we summarize some theorems for “generic” saddle-node, transcritical and pitchfork bifurcations of
\[ \dot{x} = f(x, \mu) \quad , \quad x, \mu \in \mathbb{R}. \]

Proofs use the implicit function theorem, near identity transformations and various other transformations. In all of the theorems below we assume that \( f \) has continuous (mixed) derivatives up to third order (fourth for pitchforks) for all \((\mu, x)\) near the bifurcation point \((\mu^*, x^*)\).

Saddle Node (Quadratic Tangency)

**Theorem 1** If there is a pair \((\mu^*, x^*)\) for which
\[
\begin{align*}
  f(x^*, \mu^*) &= 0 \quad (1) \\
  f_x(x^*, \mu^*) &= 0 \quad (2) \\
  f_\mu(x^*, \mu^*) &\neq 0 \quad (3) \\
  f_{xx}(x^*, \mu^*) &\neq 0 \quad (4)
\end{align*}
\]

then \( \dot{x} = f(x, \mu) \) has a saddle-node bifurcation with quadratic tangency at \((\mu^*, x^*)\).

Transcritical (2-branch)

**Theorem 2** If there is a pair \((\mu^*, x^*)\) for which
\[
\begin{align*}
  f(x^*, \mu^*) &= 0 \quad (5) \\
  f_x(x^*, \mu^*) &= 0 \quad (6) \\
  f_\mu(x^*, \mu^*) &= 0 \quad (7) \\
  f_{xx}(x^*, \mu^*) &\neq 0 \quad (8) \\
  f_{xxx}(x^*, \mu^*) &\neq 0 \quad (9)
\end{align*}
\]

then \( \dot{x} = f(x, \mu) \) has a transcritical bifurcation at \((\mu^*, x^*)\).

Pitchfork (Quadratic Tangency)

**Theorem 3** If there is a pair \((\mu^*, x^*)\) for which
\[
\begin{align*}
  f(x^*, \mu^*) &= 0 \quad (10) \\
  f_x(x^*, \mu^*) &= 0 \quad (11) \\
  f_\mu(x^*, \mu^*) &= 0 \quad (12) \\
  f_{xx}(x^*, \mu^*) &= 0 \quad (13) \\
  f_{xx}(x^*, \mu^*) &\neq 0 \quad (14) \\
  f_{xxx}(x^*, \mu^*) &\neq 0 \quad (15)
\end{align*}
\]

then \( \dot{x} = f(x, \mu) \) has a pitchfork bifurcation with quadratic tangency at \((\mu^*, x^*)\).