Definition 2.2  A bifurcation point or branch point (with respect to $\lambda$) is a solution $(y_0, \lambda_0)$ of equation (2.1) or (2.2), where the number of solutions changes when $\lambda$ passes $\lambda_0$.

TABLE 2.1. Solutions in Figure 2.3.

<table>
<thead>
<tr>
<th>$\lambda$-interval</th>
<th>Number of solutions $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda &lt; \lambda_1$</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_1 \leq \lambda &lt; \lambda_2$</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>3</td>
</tr>
<tr>
<td>$\lambda_2 &lt; \lambda &lt; \lambda_3$</td>
<td>4</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>3</td>
</tr>
<tr>
<td>$\lambda_3 &lt; \lambda &lt; \lambda_4$</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>1</td>
</tr>
<tr>
<td>Etc.</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 2.4. Deflection $v$ of a beam

2.3 Buckling and Oscillation of a Beam

Consider the following situation (see Figure 2.4). A beam is subjected at its ends to a compressive force $\Gamma$ along its axis. The beam is excited by a harmonic excitation $P(x, t)$ that depends on the spatial variable $x$ (with $0 \leq x \leq 1$) and on the time $t$. This experiment may represent a beam supporting a machine with a rotating imbalance. We denote the viscous damping by $\delta$ and the membrane stiffness by $K$. Small deflections $v(x, t)$ of the beam are described by solutions of the partial differential equation
\[ v_{xxxx} + \left( \Gamma - K \int_0^1 (v_x(\xi,t))^2 \, d\xi \right) v_{xx} + \delta v_t + v_{tt} = P \]

[Hua68], [Hol79]. Reasonable boundary conditions at \( x = 0 \) and \( x = 1 \) are \( v = v_{xx} = 0 \). We simplify the analysis by assuming perfect symmetry of \( P(x,t) \) around \( x = 0.5 \), in which case the response of the beam is likely to be in the “first mode.” That is, we take both \( P \) and \( v \) to be sinusoidal in \( x \),

\[
P(x,t) = \gamma \cos \omega t \sin \pi x,
\quad v(x,t) = u(t) \sin \pi x.
\]

The ansatz for \( P \) reflects the additional assumption of a harmonic excitation with frequency \( \omega \).
Inserting the expressions for \( P \) and the single-mode approximation of \( v \) into the PDE leads to a Duffing equation describing the temporal behavior of the displacement of the beam,

\[
\ddot{u} + \delta \dot{u} - \pi^2 (\Gamma - \pi^2) u + \frac{1}{2} K \pi^4 u^3 = \gamma \cos \omega t.
\] (2.8)

\[ u \text{ constant} \]

\[ \Gamma \]

\[ \text{stable} \]

\[ \text{stable} \]

\[ \text{unstable} \]

\[ \text{stable} \]

\textbf{Fig. 2.5.} Solutions to equation (2.8), \( \gamma = 0 \)

We first discuss the stability of deflections when the driving force is zero \( (\gamma = 0) \). It turns out (Exercise 2.4) that for \( \Gamma > \pi^2 \) there are two stable equilibria with \( u \neq 0 \), whereas for \( \Gamma < \pi^2 \) there is only one equilibrium \( (u = 0, \text{stable}) \). Here we interpret the force \( \Gamma \) as our bifurcation parameter \( \lambda \) and depict the results in a bifurcation diagram (Figure 2.5). The stationary solution \( (u, \Gamma) = (0, \pi^2) \) of equation (2.8) with \( \gamma = 0 \) is an example of a bifurcation point. As Figure 2.5 indicates, this bifurcation point separates domains of different qualitative behavior. In particular, the “trivial” solution \( u = 0 \) loses its stability at \( \Gamma = \pi^2 \). The value \( \Gamma = \pi^2 \) is called Euler’s first buckling load, because Euler calculated the critical load where a beam buckles (this will be discussed in some detail in Section 6.5).
bifurcation diagram with respect to the $\Gamma$-axis reflects the basic assumption of perfect symmetry.

So far no external energy has entered the system ($\gamma = 0$). Now we study equation (2.8) with excitation ($\gamma \neq 0$). Note that for $\gamma \neq 0$ the ODE is no longer autonomous. The possible responses of the beam can be explained most easily by discussing the experiment illustrated in Figure 2.6. Imagine a vehicle with a ball rolling inside on a cross section with one minimum ($\Gamma < \pi^2$) or two minima ($\Gamma > \pi^2$). Moving the vehicle back and forth in a harmonic fashion corresponds to the term $\gamma \cos \omega t$. We expect interesting effects in the case $\Gamma > \pi^2$. Choose (artificially)

$$\Gamma = \pi^2 + 0.2\pi^{-2}, \quad K = 16\pi^{-4}/15, \quad \gamma = 0.4, \quad \delta = 0.04.$$  

This choice leads to the specific Duffing equation

$$\ddot{u} + \frac{1}{5}\dot{u} - \frac{1}{3}u + \frac{8}{15}u^3 = \gamma \cos \omega t. \quad (2.9)$$

The harmonic responses can be calculated numerically; we defer a discussion of the specific methods to Section 6.1 and focus our attention on the results. The parameter $\Gamma$ is now kept fixed, and we choose the frequency $\omega$ as bifurcation parameter $\lambda$. This parameter is considered to be constant; it is varied in a quasi-static way. The main response of the oscillator to the sinusoidal forcing is shown in Figure 2.7. As a scalar measure of $u(t)$, amplitude $A$ is chosen. The bifurcation diagram Figure 2.7 shows a typical response of an oscillator with a hardening spring. We find two bifurcation points with parameter values

$$\omega_1 = 0.748, \quad \omega_2 = 2.502.$$  

Hysteresis effects, such as that depicted in Figure 2.7, are ubiquitous in science. Characteristic for hysteresis effects are jump phenomena, which here