Solving $2 \times 2$ linear systems of ODEs (Weiss)

In this handout we will solve $\dot{x} = Ax$, where $A = (a_{11} \ a_{12})$. In each of the three cases we will first present the general method (following the proof of JCF) and then present an example illustrating the method.

**Case I: $A$ is diagonalizable**

In this case $\lambda_1 \neq \lambda_2$ are real eigenvalues or $\lambda = \lambda_1 = \lambda_2$ with a two dimensional eigenspace. Suppose $A v_1 = \lambda_1 v_1$ and $A v_2 = \lambda_2 v_2$. Let $P = (v_1, v_2) = (v_{x1}, v_{y1}, v_{x2}, v_{y2})$. Then $P^{-1} A P = (\lambda_1 \ 0 \ 0 \ \lambda_2)$. If $x = P u$, then $\dot{x} = Ax$ is equivalent to $\dot{u} = (\lambda_1 \ 0 \ 0 \ \lambda_2) u$, with solution $u(t) = (u_1(t), u_2(t)) = (u_1(0) \exp(\lambda_1 t), u_2(0) \exp(\lambda_2 t))$. Thus $x(t) = P u(t) = (v_{x1} \ v_{y1}, v_{x2}, v_{y2}) (u_1(0) \exp(\lambda_1 t), u_2(0) \exp(\lambda_2 t)) = u_1(t)v_1 + u_2(t)v_2 = u_1(0) \exp(\lambda_1 t)v_1 + u_2(0) \exp(\lambda_2 t)v_2$.

$$x(t) = u_1(0) \exp(\lambda_1 t)v_1 + u_2(0) \exp(\lambda_2 t)v_2$$

**Example 1:** Solve $\dot{x} = Ax$, where $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The characteristic equation is $p(\lambda) = \lambda^2 - 3\lambda + 1 = 0$ with zeros $\lambda_1, \lambda_2 = (3 \pm \sqrt{5})/2$ and corresponding eigenvectors $v_1 = (-1 + (3 + \sqrt{5})/2, 1)$ and $v_2 = (-1 + (3 - \sqrt{5})/2, 1)$. Thus the solution $x(t) = u_1(0) \exp(\lambda_1 t)v_1 + u_2(0) \exp(\lambda_2 t)v_2 = u_1(0) \exp\left(\frac{3 + \sqrt{5}}{2} t\right) (-1 + (3 + \sqrt{5})/2) + u_2(0) \exp\left(\frac{3 - \sqrt{5}}{2} t\right) (-1 + (3 - \sqrt{5})/2)$.

**Case II: $A$ has double eigenvalue with one dimensional eigenspace**

In this case $\lambda_1 = \lambda_2 = \lambda$ is a double root of the characteristic equation with a one dimensional eigenspace. Choose any vector $w$ which is not an eigenvector and define $v = (A - \lambda I) w$. It follows from the Cayley-Hamilton theorem that $v$ is an eigenvector for $A$. Let $P = (v, w)$. Then $P^{-1} A P = (\lambda \ 0 \ 0 \ \lambda)$. Solving $\dot{u} = (\lambda \ 0 \ 0 \ \lambda) u$ yields $u(t) = (u_1(0) \exp(\lambda t) + u_2(0) t \exp(\lambda t), u_2(0) \exp(\lambda t))$. Thus $x(t) = P u(t)$ or

$$x(t) = u_1(0) \exp(\lambda t)v + u_2(0) t \exp(\lambda t)v + u_2(0) \exp(\lambda t)w.$$

**Example 2:** Solve $\dot{x} = Ax$, where $\begin{pmatrix} 5 & 4 \\ -1 & -1 \end{pmatrix}$. The characteristic equation is $p(\lambda) = \lambda^2 - 6\lambda + 9 = 0$ with a double zeros $\lambda = \lambda_1 = \lambda_2 = 3$. One easily sees that every eigenvector is of the form $(-x/2 \ 1)$ and hence the eigenspace is one dimensional. Choose any $w$ which is not an eigenvector, say, $w = (1 \ 1)$ and define $v = (A - \lambda I)w = \left(\begin{pmatrix} 5 & 4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right) (1 \ 1) = (6 \ -3)$. The solution is $x(t) = u_1(0) \exp(3t)(6 \ -3) + u_2(0) t \exp(3t)(6 \ -3) + u_2(0) \exp(3t)(1 \ 1)$. 

$$x(t) = u_1(0) \exp(3t)(6 \ -3) + u_2(0) t \exp(3t)(6 \ -3) + u_2(0) \exp(3t)(1 \ 1).$$
CASE III: $A$ has complex eigenvalues

In this case $\lambda_1, \lambda_2 = a \pm ib$ with $b \neq 0$. Following the proof of JCF, we choose any vector $w \neq 0$, say, $w = \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$ and define $v = \left( \begin{smallmatrix} 1/b \\ A - aI \end{smallmatrix} \right)w$. Note that $w$ and $v$ are not eigenvectors of $A$. If $P = (w, v)$, then $P^{-1}AP = \left( \begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right)$. Solving $\dot{u} = \left( \begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right)u$ yields

\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (v, w)\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (v, w) \exp(at) \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}.
\]

Example 3: Solve $\dot{x} = Ax$, where $\left( \begin{smallmatrix} -31 \\ 20 \\ -60 \\ 24 \end{smallmatrix} \right)$. The characteristic equation is $p(\lambda) = \lambda^2 + 2\lambda + 301 = 0$ with complex zeros $\lambda_1, \lambda_2 = -1 \pm 10\sqrt{3}$. We set $a = -1$ and $b = 10\sqrt{3}$. We let $w = \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$ and then $v = (1/b)(A - aI)w = (1/(10\sqrt{3})) \left( \begin{smallmatrix} -31 \\ 20 \\ -60 \\ 24 \end{smallmatrix} \right) \left( \begin{smallmatrix} -1 \\ 0 \\ 1 \\ 0 \end{smallmatrix} \right) = \left( \begin{smallmatrix} -3/\sqrt{3} \\ 6/\sqrt{3} \end{smallmatrix} \right)$. Let $P = (v, w) = \left( \begin{smallmatrix} 1 \\ 0 \\ -3/\sqrt{3} \\ 6/\sqrt{3} \end{smallmatrix} \right)$. Then

\[
x(t) = Pu(t) = \begin{pmatrix} 1 \\ 0 \\ -3/\sqrt{3} \\ 6/\sqrt{3} \end{pmatrix} \exp(-t) \begin{pmatrix} \cos((10\sqrt{3})t) \\ -\sin((10\sqrt{3})t) \\ \sin((10\sqrt{3})t) \\ \cos((10\sqrt{3})t) \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}.
\]

Note that in all cases the entries of $P$ are always real valued and all eigenvectors are real valued.

Optional remark about complex eigenvectors: In Case III there are no real valued eigenvectors but there are complex valued eigenvectors $v$ and $v^*$, where $v^*$ denotes the vector obtained by taking the complex conjugate of each element of $v$. Define real valued vectors $v_R = \left( \begin{smallmatrix} Re(v_1) \\ Re(v_2) \end{smallmatrix} \right)$ and $v_I = \left( \begin{smallmatrix} Im(v_1) \\ Im(v_2) \end{smallmatrix} \right)$. Then $v = \left( \begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix} \right) = (Re(v_1) + iIm(v_1)) = (Re(v_1)) + i(Re(v_2)) = v_R + iv_I$ and $v^* = v_R - iv_I$.

One can easily check that $x(t) = Aexp(\lambda t) + Bexp(\lambda^* t)v^*$ is an, a priori, complex solution of the system for $A, B \in \mathbb{R}$. Using the amazing fact that $\exp(i\theta) = \cos(\theta) + isin(\theta)$ we can rewrite this solution as $x(t) = Aexp(at)(\cos(bt) + isin(bt))(v_R + iv_L) + Bexp(at)(\cos(bt) - isin(bt))(v_R - iv_L)$. If we separate this solution into its real and imaginary parts, we see that the imaginary part vanishes, and we are left with the real valued solution $x(t) = Aexp(at)(\cos(bt)v_R - \sin(bt)v_I) + Bexp(at)(\cos(bt)v_I + \sin(bt)v_R)$. Is this the same solution as we obtained for Case III above?