Theorem 10 (Hopf bifurcation) Consider the two-dimensional vector field

\[ \dot{x} = f(x, \lambda), \quad (16) \]

with \( x \in \mathbb{R}^2 \) and \( \lambda \in \mathbb{R} \). Suppose (16) has a (varying) equilibrium \( x(\lambda) \) for \( \lambda \) near \( \lambda^* \) and express the eigenvalues of the Jacobian \( Df(x(\lambda), \lambda) \) as

\[ \xi_{1,2}(\lambda) = \alpha(\lambda) \pm i\beta(\lambda). \]

Then (16) is topologically equivalent to the normal form

\[ \dot{z} = (\lambda + i\omega)z + l_1(x_0, \lambda^*)|z|^2 \]

in a neighbourhood of \((x_0, \lambda^*)\), if the following conditions hold:

- (B1) \( f(x_0, \lambda^*) = 0 \), (that is, \( x(\lambda^*) = x_0 \))
- (B2) \( Df(x_0, \lambda^*) \) has a pair of purely imaginary eigenvalues \( \pm i\omega \), \( (\alpha(\lambda^*) = 0, \beta(\lambda^*) = \omega) \)
- (G1) \( l_1(x_0, \lambda^*) \neq 0 \), where \( l_1 \) is specified below,
- (G2) \( \frac{d}{d\lambda} \alpha(\lambda^*) \neq 0 \).

Remark 11 We call \( l_1(x_0, \lambda^*) \) the first Lyapunov quantity. It can be computed from \( f \) in (16). If you need it you should look it up in the literature.

The sign of the first Lyapunov quantity determines the stability properties of the periodic orbit that appears in the bifurcation. We have the following overview:

- **"Supercritical"**
  - \( l_1(x_0, \lambda^*) < 0 \)
- **"Harmonic oscillator"**
  - \( l_1(x_0, \lambda^*) = 0 \)
- **"Subcritical"**
  - \( l_1(x_0, \lambda^*) > 0 \)

\[ \lambda < 0 \quad \lambda = 0 \quad \lambda > 0 \]