Homework due and reading quiz on 02/14/19

1. Read Chapter 13 (HIV) of Sompayrac

2. A simple model for the population density of T-cells is

\[
\frac{dT}{dt} = s - dT + aT(1 - T/T_{\text{max}}).
\]

Here \(s\) is the rate at which new T-cells are produced, \(d\) is their death rate, \(a\) is the maximum proliferation rate, and \(T_{\text{max}}\) is the cell density at which the proliferation stops.

(a) Using the nondimensionalized version of this ODE that you found in HW2, find the equilibrium points and use the Method of Linearization to determine their stability.

(b) Suppose \(T_{\text{max}} = 1200 \text{ mm}^3/\text{day}, a = 0.5/\text{day}, d = 0.01/\text{day}\) and \(s = 5.0/\text{day mm}^3\). Calculate the equilibrium cell densities.

3. A simplified and nondimensionalized version of the classic infection model assumes that the virus density is at equilibrium, and can be written as

\[
\frac{dx}{d\tau} = 1 - x - R_0xy \\
\frac{dy}{d\tau} = R_0xy - y
\]

(a) Find the equilibrium points and use the Method of Linearization to determine their stability.

4. A particular two species system can be modeled by

\[
\frac{dP}{dt} = r_1P(1 - P/K) - A \frac{PQ}{D + P} \\
\frac{dQ}{dt} = r_2Q \left(1 - \frac{Q}{P}\right)
\]

(a) Using the nondimensionalized version of this ODE that you found in HW2, find the equilibrium points and use the Method of Linearization to determine their stability.

5. Consider the classical in-host model of viral infection.

(a) Why do infected cells survive for time \(1/a\)?

(b) Each infected cell produces virus at rate \(k\), and while producing virus, infected cells live for time \(1/a\). Thus each infected cell produces \(k/a\) virions. Each virion survives for time \(1/u\), and assuming that the target cell population remains at \(\lambda/d\), then each viron infects how many cells?

(c) Find a formula for the number of cells infected by \(k/a\) virions released from one infected cell at the beginning of the infection?

6. Consider the classical in-host model of viral infection. Suppose that at some time a drug is administered that instantaneously stops all new infections. Use the model to describe what follows with as much detail as possible.

7. Slightly modify the classical in-host model of viral infection by replacing the first ODE by \(\frac{dT}{dt} = s - dT + aT(1 - T/T_{\text{max}})\).
(a) Why might you wish to do this?
(b) Nondimensionalize the modified system?
(c) Find all the equilibrium points and determine their stability?
(d) Are there any bifurcations?
(e) As far as equilibrium points and their stability, describe the differences with the classical in-host model?

8. Consider the classical in-host model of viral infection with infection occurring at time 0 with a single virion. Also assume that $R_0 > 1$ and that for a short time after infection, there is little change of the large uninfected cell population.

(a) Show that there is a linear system of two ODEs that should approximate the $y(t)$ and $v(t)$ terms in the classical in-host model well for short time.
(b) Verify that the solution $v(t)$ of the linear system grows exponentially. Find an expression for the exponent.
(c) Conclude that for the classical in-host model of viral infection, if $R(0) > 1$ then at the onset of an infection, the number of virions grows exponentially.
(d) Can you conclude the same about the number of infected cells? (explain!)

9. Recall the SIR infectious disease transmission model

$$\frac{dS}{dt} = -bSI$$
$$\frac{dI}{dt} = bSI - cI$$
$$\frac{dR}{dt} = cI.$$ 

- We nondimensionalized by setting $x = S/S(0), y = I/S(0), z = R/S(0)$ and obtained the equivalent system

$$\frac{dx}{d\tau} = -R_0 xy$$
$$\frac{dy}{d\tau} = R_0 xy - y$$
$$\frac{dz}{d\tau} = y,$$

where $R_0 = S(0)b/c$. Let’s now consider an alternative scaling.

(a) Show that any solution $(S(t), I(t), R(t))$ satisfies $S(t) + I(t) + R(t)$ is constant in time. Hint: Add the three left hand sides of the system of three ODEs. The constant is $S(0) + I(0) + R(0) \equiv N$.
(b) Nondimensionalize the the original SIR system by setting $x = S/N, y = I/N, z = R/N$.
(c) Compare with the previous nondimensionalization.

10. Additional problems TBA