Geometrical and computational aspects

(reverse the arrows) the limit cycle is unstable. A semi-stable limit cycle can also occur, in which the paths approach on one side and recede on the other.

Plotting a limit cycle may be difficult. Both the limit cycle and the neighbouring paths can diverge very considerably from the idealized pattern of Fig. 3.17; for example Fig. 3.23 shows the limit cycle for van der Pol’s equation with a moderate value for the parameter. In general there is no way of finding a single point, exactly lying on the cycle, from which to start; the cycle has to be located by ‘squeezing’ it between inner and outer spirals. Clearly it is helpful to reverse the sign of (or of in the program input) if necessary, so as to approach rather than recede from the cycle during the process of locating it.

Exercises

1. By considering the variation of path direction on closed curves round the equilibrium points, find the index in each case of Fig. 3.24.

2. The motion of a damped pendulum is described by the equations
   \[ \ddot{\theta} = \omega, \quad \dot{\omega} = -k\omega - \nu^2 \sin \theta, \]
   where \( k (>0) \) and \( \nu \) are constants. Find the indices of all the equilibrium states.

3. Find the index of the equilibrium points of the following systems: (i) \( \dot{x} = 2xy, \quad \dot{y} = 3x^2 - y^2 \); (ii) \( \dot{x} = y^2 - x^4, \quad \dot{y} = x^3y \); (iii) \( \dot{x} = x - y, \quad \dot{y} = x - y^2 \).

4. For the linear system \( \dot{x} = ax + by \), index at the origin by evaluating eq: \( -bc \). (Hint: choose \( \Gamma \) to be the ellip...

5. The equation of motion of a bar attracted by a parallel current-carrying force is
   \[ \ddot{x} + c(x - x_0) \]
   where \( c, a, \) and \( \lambda \) are positive constants and find the indices of the equilibrium of the phase diagram.

6. Show that the equation
   \[ \ddot{x} - \epsilon(1 - x) \]
   has an equilibrium point of index 1 \( x = y \). (It also has a limit cycle, \( x \) radius \( a \) to show that, for all \( a \),
   \[ \int_0^x \frac{dz}{1 - 2\epsilon(1 - a^2) \sin \theta} \]

7. A limit cycle encloses \( N \) nodes, \( F \) all of the type of Chapter 2. Show that

8. Given the system
   \[ \dot{x} = X(x, y) \cos \alpha - Y(x, y) \sin \alpha \]
   where \( \alpha \) is a parameter, prove that the limit cycle is not a limit cycle (Chapter 2).

Fig. 3.24.
2. Nonlinear Systems: Local Theory

(c) \[ \begin{align*}
\dot{x} &= -y + x^5 \\
\dot{y} &= x + y^2 
\end{align*} \]

2. Let

\[ f(x) = \begin{cases} 
\frac{x}{\ln |x|} & \text{for } x \neq 0 \\
0 & \text{for } x = 0 
\end{cases} \]

Show that \( f'(0) = \lim_{x \to 0} f'(x) = 0 \); i.e., \( f \in C^1(\mathbb{R}) \), but that \( f''(0) \) is undefined.

3. Show that if \( x = 0 \) is a zero of the function \( f: \mathbb{R} \to \mathbb{R} \) and \( f \in C^1(\mathbb{R}) \) then

\[ f(x) = Df(0)x + F(x) \]

where \( |F(x)/x| \to 0 \) as \( x \to 0 \). Show that this same result holds for \( f: \mathbb{R}^2 \to \mathbb{R}^2 \). Hint: Use Taylor’s Theorem and the continuity of \( Df(x) \).

4. Determine the nature of the critical points of the following nonlinear systems (Cf. Problem 1 in Section 2.6); be as specific as possible.

(a) \[ \begin{align*}
\dot{x} &= x - xy \\
\dot{y} &= y - x^2 
\end{align*} \]

(b) \[ \begin{align*}
\dot{x} &= -4x + 2xy - 8 \\
\dot{y} &= 4y^2 - x^2 
\end{align*} \]

(c) \[ \begin{align*}
\dot{x} &= 2x - 2xy \\
\dot{y} &= 2y - x^2 + y^2 
\end{align*} \]

(d) \[ \begin{align*}
\dot{x} &= -x^2 - y^2 + 1 \\
\dot{y} &= 2x 
\end{align*} \]

(e) \[ \begin{align*}
\dot{x} &= x^2 - y^2 + 1 \\
\dot{y} &= 2xy 
\end{align*} \]

(f) \[ \begin{align*}
\dot{x} &= x^2 - y^2 - 1 \\
\dot{y} &= 2y 
\end{align*} \]

2.11 Nonhyperbolic Critical Points in \( \mathbb{R}^2 \)

In this section we present some results on nonhyperbolic critical points of planar analytic systems. This work originated with Poincaré [P] and was extended by Bendixon [B] and more recently by Andronov et al. [A–I]. We assume that the origin is an isolated critical point of the planar system

\[ \begin{align*}
\dot{x} &= P(x, y) \\
\dot{y} &= Q(x, y) 
\end{align*} \]  (1)

where \( P \) and \( Q \) are analytic in some neighborhood of the origin. In Sections 2.9 and 2.10 we have already presented some results for the case when the matrix of the linear part \( A = Df(0) \) has pure imaginary eigenvalues, i.e., when the origin is a center of the linearized system. In this section we give some results established in [A–I] for the case when the matrix \( A \) has one or two zero eigenvalues, but \( A \neq 0 \). And these results are extended to higher dimensions in Section 2.12.

First of all, note that if \( P \) and \( Q \) begin with \( m \)-th degree terms \( P_m \) and \( Q_m \), then it follows from Theorem 2 in Section 2.10 that if the function

\[ g(\theta) = P_m(\cos \theta, \sin \theta) \]

is not identically zero, then there are at most \( 2(m + 1) \) directions \( \theta = \theta_0 \) along which a trajectory of (1) may approach the origin. These directions are given by solutions of the equation \( g(\theta) = 0 \). Suppose that \( g(\theta) \) is not identically zero, then the solution curves of (1) which approach the origin along these tangent lines divide a neighborhood of the origin into a finite number of open regions called sectors. These sectors will be of one of three types as described in the following definitions; cf. [A–I] or [L]. The trajectories which lie on the boundary of a hyperbolic sector are called separatrices. Cf. Definition 1 in Section 3.11.

**Definition 1.** A sector which is topologically equivalent to the sector shown in Figure 1(a) is called a **hyperbolic sector**. A sector which is topologically equivalent to the sector shown in Figure 1(b) is called a **parabolic sector**. And a sector which is topologically equivalent to the sector shown in Figure 1(c) is called an **elliptic sector**.

![Figure 1](image-url)

Figure 1. (a) A hyperbolic sector. (b) A parabolic sector. (c) An elliptic sector.

In Definition 1, the homeomorphism establishing the topological equivalence of a sector to one of the sectors in Figure 1 need not preserve the direction of the flow; i.e., each of the sectors in Figure 1 with the arrows reversed are sectors of the same type. For example, a saddle has a deleted neighborhood consisting of four hyperbolic sectors and four separatrices. And an improper node has a deleted neighborhood consisting of one parabolic sector. According to Theorem 2 below, the system

\[ \dot{x} = y \]
2. Nonlinear Systems: Local Theory

\[ \dot{y} = -x^3 + 4xy \]

has an elliptic sector at the origin; cf. Problem 1 below. The phase portrait for this system is shown in Figure 2. Every trajectory which approaches the origin does so tangent to the \( x \)-axis.

A deleted neighborhood of the origin consists of one elliptic sector, one hyperbolic sector, two parabolic sectors, and two separatrices. This type of critical point is called a critical point with an elliptic domain; cf. \([A-1]\).

Figure 2. A critical point with an elliptic domain at the origin.

Another type of nonhyperbolic critical point for a planar system is a saddle-node. A saddle-node consists of two hyperbolic sectors and one parabolic sector (as well as three separatrices and the critical point itself). According to Theorem 1 below, the system

\[ \begin{align*}
\dot{x} &= x^2 \\
\dot{y} &= y
\end{align*} \]

has a saddle-node at the origin; cf. Problem 2. Even without Theorem 1, this system is easy to discuss since it can be solved explicitly for \( x(t) = \)

2.11. Nonhyperbolic Critical Points in \( R^2 \)

\((1/x_0 - t)^{-1}\) and \( y(t) = y_0 e^t \). The phase portrait for this system is shown in Figure 3.

Figure 3. A saddle-node at the origin.

One other type of behavior that can occur at a nonhyperbolic critical point is illustrated by the following example:

\[ \begin{align*}
\dot{x} &= y \\
\dot{y} &= x^2
\end{align*} \]

The phase portrait for this system is shown in Figure 4. We see that a deleted neighborhood of the origin consists of two hyperbolic sectors and two separatrices. This type of critical point is called a cusp.

As we shall see, besides the familiar types of critical points for planar analytic systems discussed in Section 2.10, i.e., nodes, foci, topological saddles and centers, the only other types of critical points that can occur for (1) when \( A \neq 0 \) are saddle-nodes, critical points with elliptic domains and cusps.

We first consider the case when the matrix \( A \) has one zero eigenvalue, i.e., when \( \det A = 0 \), but \( \text{tr} A \neq 0 \). In this case, as in Chapter 1 and as is
shown in [A–I] on p. 338, the system (1) can be put into the form

\[
\begin{align*}
\dot{x} &= p_2(x, y) \\
\dot{y} &= y + q_2(x, y)
\end{align*}
\]  

(2)

where \( p_2 \) and \( q_2 \) are analytic in a neighborhood of the origin and have expansions that begin with second-degree terms in \( x \) and \( y \). The following theorem is proved on p. 340 in [A–I]. Cf. Section 2.12.

**Theorem 1.** Let the origin be an isolated critical point for the analytic system (2). Let \( y = \phi(x) \) be the solution of the equation \( y + q_2(x, y) = 0 \) in a neighborhood of the origin and let the expansion of the function \( \psi(x) = p_2(x, \phi(x)) \) in a neighborhood of \( x = 0 \) have the form \( \psi(x) = a_m x^m + \cdots \) where \( m \geq 2 \) and \( a_m \neq 0 \). Then (1) for odd \( m \) and \( a_m > 0 \), the origin is an unstable node, (2) for odd \( m \) and \( a_m < 0 \), the origin is a topological saddle and (3) for even \( m \), the origin is a saddle-node.

Next consider the case when \( A \) has two zero eigenvalues, i.e., \( \det A = 0 \), \( \text{tr} A = 0 \), but \( A \neq 0 \). In this case it is shown in [A–I], p. 356, that the system (1) can be put in the “normal” form

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= a_k x [1 + h(x)] + b_n x^n y [1 + g(x)] + y^2 R(x, y)
\end{align*}
\]  

(3)

where \( h(x), g(x) \) and \( R(x, y) \) are analytic in a neighborhood of the origin, \( h(0) = g(0) = 0 \), \( k \geq 2 \), \( a_k \neq 0 \) and \( n \geq 1 \). The next two theorems are proved on pp. 357–362 in [A–I]. Cf. Section 2.13.

**Theorem 2.** Let \( k = 2m + 1 \) with \( m \geq 1 \) in (3) and let \( \lambda = b_n^2 + 4 (m + 1) a_k \).

Then if \( a_k > 0 \), the origin is a topological saddle. If \( a_k < 0 \), the origin is

(1) a focus or a center if \( b_n = 0 \) and also if \( b_n \neq 0 \) and \( n > m \) or \( n = m \)

and \( \lambda < 0 \), (2) a node if \( b_n \neq 0 \), \( n \) is an even number and \( n < m \) and also \( b_n \neq 0 \), \( n \) is an odd number, \( n = m \) and \( \lambda > 0 \) and (3) a critical point with an elliptic domain if \( b_n \neq 0 \), \( n \) is an odd number and \( n < m \) and also \( b_n \neq 0 \), \( n \) is an even number, \( n = m \) and \( \lambda > 0 \).

**Theorem 3.** Let \( k = 2m \) with \( m \geq 1 \) in (3). Then the origin is (1) a node if \( b_n = 0 \) and also if \( b_n \neq 0 \) and \( n = m \) and (2) a saddle-node if \( b_n \neq 0 \) and \( n < m \).

We see that if \( \text{DF}(x_0) \) has one zero eigenvalue, then the critical point \( x_0 \)

is either a node, a topological saddle, or a saddle-node; and if \( \text{DF}(x_0) \) has two zero eigenvalues, then the critical point \( x_0 \) is either a focus, a center, a node, a topological saddle, a saddle-node, a cusp, or a critical point with an elliptic domain.

Finally, what if the matrix \( A = 0? \) In this case, the behavior near the origin can be very complex. If \( P \) and \( Q \) begin with \( m \)-th degree terms, then the separatrices may divide a neighborhood of the origin into \( 2(m + 1) \)

sectors of various types. The number of elliptic sectors minus the number of hyperbolic sectors is always an even number and this number is related to the index of the critical point discussed in Section 3.12 of Chapter 3.

For example, the homogenous quadratic system

\[
\begin{align*}
\dot{x} &= x^2 + xy \\
\dot{y} &= \frac{1}{2} y^2 + xy
\end{align*}
\]

has the phase portrait shown in Figure 5. There are two elliptic sectors and two parabolic sectors at the origin. All possible types of phase portraits for homogenous, quadratic systems have been classified by the Russian mathematician L.S. Lyapunov [19]. For more information on the topic, cf. the book by Nemyski and Stepanov [N/S].

**Remark.** A critical point, \( x_0 \), of (1) for which \( \text{DF}(x_0) \) has a zero eigenvalue is often referred to as a multiple critical point. The reason for this is made clear in Section 4.2 of Chapter 4 where it is shown that a multiple
critical point of (1) can be made to split into a number of hyperbolic critical points under a suitable perturbation of (1).

Figure 5. A nonhyperbolic critical point with two elliptic sectors and two parabolic sectors.

Problem Set 11

1. Use Theorem 2 to show that the system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x^2 + 4xy
\end{align*}
\]

has a critical point with an elliptic domain at the origin. Note that \(y = x^2/(2 \pm \sqrt{2})\) are two invariant curves of this system which bound two parabolic sectors.

2. Use Theorem 1 to determine the nature of the critical point at the origin for the following systems:

(a) \(\dot{x} = x^2\) \\
(b) \(\dot{x} = y, \dot{y} = x\)

(b) \(\dot{x} = x^2 + 2xy + y^2\)

2.12 Center Manifold Theory

In Section 2.8 we presented the Hartman–Grobman Theorem, which showed that, in a neighborhood of a hyperbolic critical point \(x_0 \in E\), the nonlinear system

\[
\mathbf{x} = f(x) \tag{1}
\]

is topologically conjugate to the linear system

\[
\mathbf{x} = Ax \tag{2}
\]

with \(A = DF(x_0)\), in a neighborhood of the origin. The Hartman–Grobman Theorem therefore completely solves the problem of determining the stability and qualitative behavior in a neighborhood of a hyperbolic critical point.
Def (Temporary) Prop P is *linearly, structurally stable* if

\[ \text{Given that } P \text{ is true for solns of } x = Ax, \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]

then \( \exists \varepsilon > 0 \) for any \( \overline{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) where

\( \alpha = -1, \beta = 1, \gamma = 1, \delta = 1 \). The property P

is also true for solns of \( x = \overline{A}x \).

**Example**

\[ W^s(0,0) = IR^2 \]

1. \( P = \{ \text{all solns approach } (0,0) \text{ as } t \to -\infty \} \) is \( \text{LSS} \).
2. \( P = \{ (1,0) \text{ is a saddle point} \} \) is \( \text{LSS} \).
3. \( P = \{ (0,0) \text{ is a stable node} \} \) is \( \text{LSS} \).
4. \( P = \{ (1,0) \text{ is a center} \} \) is \( \text{not LSS} \).
5. \( P = \{ (0,1) \text{ is a degenerate node} \} \) is \( \text{not LSS} \).

**General Notions of Stability**

\[ x = f(x), \quad x \in C^1 \]

Assume \( 0 \in \text{cp} \)

**Def 1** \( x = (0,0) \) is an **attracting cp** if \( \exists \delta > 0 \)

\[ \lim_{t \to -\infty} x(t) = 0 \quad \text{whenever} \quad x(0) \in B(0,\delta) \]

All orbits that start near \( (0,0) \) approaches \( (0,0) \) as \( t \to -\infty \)

\( 0 \), stable nodes (including star and degenerate) are stable saddles

**Def 2** \( x = (0,0) \) is globally **attracting cp** if

\( W^s(0,0) = IR^2 \)

or save list as above
2. \[ y'' = y' A + \frac{1}{2} A^2 - \frac{3}{2} A^3 \]

Always CS
map smoothly bounded, \( y' \neq y \)

Proof Consider map \( (c, g) \rightarrow (c', g') \)
\[ \theta = \tan^{-1} \left( \frac{\sin \theta}{\cos \theta} \right) \]

**Def.** \( x = (0, 1, 0) \) is **Lyapunov stable** if \( A \gg 0 \) & \( S > 0 \)

\[ \| x(0) \| < S \Rightarrow \| x(t) \| < A + \varepsilon \]

All trajectories starting sufficiently close to \((0, 1, 0)\) remain close to \((0, 1, 0)\) at later time.

**Ex.** Same list as above + centers

**Results**

1. Centers are Lyap stable but not attracting.
2. \( x = 1 - 3.14 \theta \quad \theta = \frac{\pi}{2} \) (globally) attracting but not Lyap stable.

**Def.** \( x = (0, 1, 0) \) is asymptotic stable if it is Lyap stable &

not globally stable.

Stable node, stable spiral.

**Def.** \( x = (0, 1, 0) \) is neutrally stable if it is Lyap stable but not attracting center.

**Def.** \( x = (0, 1, 0) \) is unstable if it is neither attracting or Lyap stable.
\[ \dot{x} = f(x) \quad f: \mathbb{R}^n \to \mathbb{R}^n \text{ smooth} \]
\[ \text{Spie } f'(q) = 0 \quad \text{a cp} \]

Consider linearized system at \( x = a \):
\[ \dot{x} = (Df(a))x \quad \text{we know everything about this linear system.} \]

Is the behavior of orbits near \( x = a \) for the non-linear system "similar" to orbits for the linearized system?

In general, NO

**Problem #2**
\[ \begin{align*}
\dot{x} &= -y - x \sqrt{x^2 + y^2} \\
\dot{y} &= x - y \sqrt{x^2 + y^2}
\end{align*} \]
(0,0) cp "stable spiral" (globally attracting)
(0,0) is a center of linearized system

Possible notion of "similar"

**Theorem** If \( Df(a) \) has \( n \)-distinct eigenvalues with negative real part (\( n = 2 \): stable node, stable spiral)
then \( x = a \) is asymptotically stable.
Consequence of

Thm (Poincaré–Lyapunov). [Verhulst Thm 7.1]

Consider \( \dot{x} = Ax + B(t) x + f(t, x) \).

1) Eigenvalues of \( A \) have neg real part

2) \( B(t) \) is an \( n \times n \) matrix, \( \lim_{t \to \infty} ||B(t)|| = 0 \)

3) \( f(t, x) \) is \( C^1 \), \( \|f(t, x)\| \leq \|x\| \) and \( \|x(t)\| = 0 \) uniformly in \( t \)

\( (\Rightarrow x = 0 \text{ is a soln}) \).

Then \( \exists C, \theta, \delta, \mu > 0 \) so \( \|x(\theta)\| \leq \|x(0)\| e^{-\delta(\theta-t)} \)

\( \|x(t)\| \leq C \|x(0)\| e^{-\mu t} \) for \( \|x(0)\| < \delta \)

Gronwall's Lemma

Integral form of ODE

Estimate of \( \|x(t)\| \) for \( x = Ax \)