contrary. Then by 6.5G there exists some \( \epsilon > 0 \) such that \( A \) contains no finite \( \epsilon \)-dense subset. Thus if \( x_k \in A \), then the set \( \{ x_k \} \) is not \( \epsilon \)-dense in \( A \), so there exists \( x_n \in A \) such that \( \rho(x_1, x_n) > \epsilon \). But then \( \{ x_n \}_{n=1}^\infty \) is not \( \epsilon \)-dense in \( A \) and so there exists \( x_1 \in A \) such that \( \rho(x_1, x_n) > \epsilon \) and \( \rho(x_2, x_n) > \epsilon \). Continuing in this fashion we may construct a sequence \( \{ x_n \}_{n=1}^\infty \) of points of \( A \) such that \( \rho(x_j, x_k) > \epsilon \) for any \( j, k \in I \). (Since \( \rho \) has no Cauchy subsequence, which contradicts our hypothesis. This contradiction shows that \( A \) must be totally bounded and the proof is complete.

Thus, if \( n \) is such a subset \( E \) of \( I \) defined in 6.3B is not totally bounded. For, since \( \rho(x_j, x_k) = \sqrt{2} \) if \( j \neq k \), the sequence \( x_1, x_2, \ldots \) has no Cauchy subsequence.

**Exercises 6.3**

1. Prove that every bounded subset of \( R^2 \) is totally bounded.
2. Give an example of a bounded subset of \( I^\infty \) which is not totally bounded.
3. Give an example of an infinite subset of \( I^2 \) which is totally bounded.
4. Prove that every finite subset of a metric space \( M \) is totally bounded.
5. If \( \langle M, \rho \rangle \) is totally bounded and \( A \subseteq M \), prove that \( \langle A, \rho \rangle \) is totally bounded.
6. Let \( B \) be a subset of the metric space \( M \). Prove that \( B \) is dense in \( M \) if and only if \( B \) is \( \epsilon \)-dense in \( M \) for every \( \epsilon > 0 \).
7. Let \( A \) be a finite bounded subset of \( R^1 \). Prove that there is at least one cluster point of \( A \) in \( R^1 \). (Hint: Suppose \( A \subseteq J \), where \( J \) is a closed bounded interval. If \( J \) is divided in half, then at least one of the halves must contain infinitely many points of \( A \). Call this half \( J \). Continue this process.) This result is known as the Bolzano-Weierstrass theorem.
8. Give another proof of the Bolzano-Weierstrass theorem, beginning as follows: Let \( A \) be an infinite bounded subset of \( R^1 \). Let \( \{ a_n \}_{n=1}^\infty \) be a sequence of distinct points of \( A \). Then \( \{ a_n \}_{n=1}^\infty \) contains a Cauchy subsequence (why?). Now finish the proof.

### 6.4 COMPLETE METRIC SPACES

In 2.10D we saw that in the metric space \( R^1 \) every Cauchy sequence of points in \( R^1 \) converges to a point in \( R^1 \). In 4.3F we noted that there are metric spaces \( \langle M, \rho \rangle \) in which not all Cauchy sequences of points of \( M \) converge to a point in \( M \).

**6.4A. Definition.** We say that the metric space \( M \) is complete if every Cauchy sequence of points in \( M \) converges to a point in \( M \).

Thus \( R^1 \) is complete by 2.10D. In the exercises you will be asked to show that \( R^2 \) and \( R_4 \) are also complete.

**6.4B.** We now show that \( I^2 \) is complete. If \( s = (s^{(1)}, s^{(2)}, \ldots) \) is a Cauchy sequence of points in \( I^2 \), we must find \( s \in I^2 \) such that \( s^{(n)} \to s \) as \( n \to \infty \). Since each \( s^{(n)} \) is itself a sequence, the notation will be a little complicated. Denote the \( k \)th term of the sequence \( s^{(n)} \) by \( s_k^{(n)} \) so that

\[
s^{(n)} = \{ s_k^{(n)} \}_{k=1}^\infty \quad \text{and} \quad \| s^{(n)} \|_2 = \sum_{k=1}^\infty s_k^{(n)^2}.
\]
Since $s^{(1)}, s^{(2)}, \ldots$ is a Cauchy sequence in $\ell^2$, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that
\[ d[s^{(n)}, s^{(m)}] < \epsilon \quad (n, m \geq N). \]
That is,
\[ \|s^{(n)} - s^{(m)}\|_2 < \epsilon \quad (n, m \geq N). \]
which implies
\[ \|s^{(n)} - s^{(N)}\|_2 < \epsilon \quad (n \geq N). \]
Thus, if $n \geq N$,
\[ \|s^{(n)}\|_2 = \|s^{(n)} - s^{(N)} + s^{(N)}\|_2 < \epsilon + \|s^{(N)}\|_2. \]
Thus, for some $A > 0$,
\[ \|s^{(n)}\|_2 < A \quad (n \geq N). \]
Now, for any $k \in I$, we have from (1)
\[ |s_k^{(n)} - s_k^{(m)}| < \|s^{(n)} - s^{(m)}\|_2 < \epsilon \quad (n, m \geq N). \]
Hence (for fixed $k$) the sequence $(s_k^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in $R^1$ and so, by 2.10D, converges to a number $s_k \in R^1$. Let $s$ denote the sequence $(s_k)_{k=1}^{\infty}$. First we will show that $s \in \ell^2$. From (2) we have
\[ \sum_{k=1}^{\infty} s_k^{(n)} < A^2 \quad (n \geq N). \]
Hence for any integer $L \in I$,
\[ \sum_{k=1}^{L} s_k^{(n)} < A^2 \quad (n \geq N). \]
But for $k = 1, 2, \ldots, L$, we have $s_k^{(n)} \rightarrow s_k$ as $n \rightarrow \infty$. Hence letting $n \rightarrow \infty$ in (3) and using 2.7A and 2.7E, we have
\[ \sum_{k=1}^{L} s_k^2 < A^2 \quad (L = 1, 2, \ldots). \]
It follows that
\[ \sum_{k=1}^{\infty} s_k^2 < A^2, \]
which proves that $s = (s_k)_{k=1}^{\infty}$ is in $\ell^2$. From (1) we have
\[ \sum_{k=1}^{\infty} (s_k^{(n)} - s_k^{(m)})^2 < \epsilon^2 \quad (n, m \geq N). \]
Hence for $L \in I$,
\[ \sum_{k=1}^{L} (s_k^{(n)} - s_k^{(m)})^2 < \epsilon^2 \quad (n, m \geq N). \]
Letting $m \rightarrow \infty$ (and using $\lim_{m \rightarrow \infty} s_k^{(m)} = s_k$, 2.7A, and 2.7E), we have
\[ \sum_{k=1}^{L} (s_k^{(n)} - s_k)^2 < \epsilon^2 \quad (n \geq N; L \in I), \]
and so
\[ \sum_{k=1}^{\infty} (s_k^{(n)} - s_k)^2 < \epsilon^2 \quad (n \geq N). \]
But this says that \( \rho(s^{(n)}, s) = \|s^{(n)} - s\|_2 < \epsilon \) if \( n \geq N \), which proves that \( s^{(1)}, s^{(2)}, \ldots \) converges in \( L^2 \) to the point \( s \). This completes the proof.

6.4C. Theorem. If \( \langle M, \rho \rangle \) is a complete metric space and \( A \) is a closed subset of \( M \), then \( \langle A, \rho \rangle \) is also complete.

**Proof:** Let \( \{x_n\}_{n=1}^{\infty} \) be a Cauchy sequence of points in \( \langle A, \rho \rangle \). We must show that \( \{x_n\}_{n=1}^{\infty} \) converges to a point in \( A \). Since \( A \subseteq M \), \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence of points of \( M \). Thus since \( M \) is complete, \( \{x_n\}_{n=1}^{\infty} \) must converge to some \( x \in M \). But \( x \) is a limit point of \( A \) because \( x \) is the limit of a sequence of points in \( A \). Hence \( x \in A \) because \( A \) is closed, and the proof is complete.

Thus the metric space \([0, 1]\) (with absolute value metric) is complete. For \([0, 1]\) is a closed subset of \( \mathbb{R}^1 \).

Here is a generalization of the nested interval theorem 2.10E.

6.4D. Theorem. Let \( \langle M, \rho \rangle \) be a complete metric space. For each \( n \in I \) let \( F_n \) be a non-empty, closed bounded subset of \( M \) such that

(a) \( F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq F_{n+1} \supseteq \cdots \)

and

(b) \( \text{diam } F_n \to 0 \) as \( n \to \infty \).

Then \( \bigcap_{n=1}^{\infty} F_n \) contains precisely one point.

**Proof:** For each \( n \in I \), let \( a_n \) be any point in \( F_n \). Then, by (a),

\[
a_n, a_{n+1}, a_{n+2}, \ldots \quad \text{all lie in } F_n.
\]

Given \( \epsilon > 0 \) there exists, by (b), an integer \( N \in I \) such that \( \text{diam } F_N < \epsilon \). Now \( a_N, a_{N+1}, a_{N+2}, \ldots \) all lie in \( F_N \). For \( m, n \geq N \) we then have \( \rho(a_n, a_m) < \text{diam } F_N < \epsilon \). This proves that \( \{a_n\}_{n=1}^{\infty} \) is a Cauchy sequence. Since \( M \) is complete there exists \( a \in M \) such that \( \lim_{n \to \infty} a_n = a \). For any \( n \in I \), the statement (1) then shows that \( a \) is a limit point of the closed set \( F_n \) and hence \( a \in F_n \). Thus \( a \in \bigcap_{n=1}^{\infty} F_n \). If \( b \in M, b \neq a \), then \( \rho(a, b) > \text{diam } F_n \) for \( n \) sufficiently large. Hence \( b \) cannot be in \( \bigcap_{n=1}^{\infty} F_n \). This completes the proof.

6.4E. We are now going to discuss a class of functions called contractions. Although their usefulness will not be immediately apparent, they turn out to have important applications. In a later chapter we use a result on contractions to prove an existence theorem for differential equations.

To simplify notation in this discussion, if \( T: M \to M \) and if \( x \in M \), we will write \( Tx \) instead of \( T(x) \). We will also write \( T^2 \) instead of \( T \circ T \), \( T^3 \) instead of \( T \circ T \circ T \), etc.

**Definition.** Let \( \langle M, \rho \rangle \) be a metric space. If \( T: M \to M \), we say that \( T \) is a contraction on \( M \) if there exists \( \alpha \in R \) with \( 0 < \alpha < 1 \) such that

\[
\rho(Tx, Ty) < \alpha \rho(x, y) \quad (x, y \in M).
\]

(We emphasize that the number \( \alpha \) must be independent of \( x \) and \( y \)).
Thus $T$ is a contraction if the distance from $Tx$ to $Ty$ is not greater than $\alpha$ times the distance from $x$ to $y$. We see that applying $T$ to each of two points "contracts" the distance between them.

The reader should verify that if $T$ is a contraction on $M$ then $T$ is continuous on $M$.

Here is an easy example of a contraction. If $u = \{u_n\}_{n=1}^{\infty} \in F^2$, let $Tu = (u_n/2)_{n=1}^{\infty}$. Then $T$ is a contraction on $F^2$. For if $v = \{v_n\}_{n=1}^{\infty}$ is any other point in $F^2$, then

$$\rho(Tu, Tv) = \|Tu - Tv\|_2 = \left( \sum_{n=1}^{\infty} \left( \frac{u_n}{2} - \frac{v_n}{2} \right)^2 \right)^{1/2} = \frac{1}{2} \|u - v\|_2.$$

Thus in this example, $\alpha$ may be taken to be $\frac{1}{2}$. For this $T$ it is obvious that there is one and only one sequence $s \in F^2$ such that $Ts = s$—namely, the sequence $0, 0, 0, \ldots$. This illustrates the following theorem, which is called the Picard or the Banach fixed-point theorem.

**6.4F. Theorem.** Let $\langle M, \rho \rangle$ be a complete metric space. If $T$ is a contraction on $M$, then there is one and only one point $x$ in $M$ such that $Tx = x$. (This is often stated as "$T$ has precisely one fixed point.")

**Proof:** Suppose $x, y \in M$. We have $\rho(Tx, Ty) \leq \alpha \rho(x, y)$ for some $\alpha, 0 < \alpha < 1$. Then $\rho(T^2x, T^2y) \leq \alpha \rho(Tx, Ty) \leq \alpha^2 \rho(x, y)$. Indeed, for any $n \in \mathbb{N}$, it is easy to show that

$$\rho(T^n x, T^n y) \leq \alpha^n \rho(x, y) \quad (x, y \in M). \quad (1)$$

Now choose any $x_0 \in M$. Let $x_1 = Tx_0, x_2 = Tx_1, \ldots, x_{n+1} = Tx_n$. Then $x_2 = T^2x_0$ and, for any $n \in \mathbb{N}, x_n = T^nx_0$. We will first show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. For if $m, n \in \mathbb{N}$ (and $m > n$, say, so that $m = n + p$) we have

$$\rho(x_n, x_m) = \rho(x_n, x_{n+p}) \leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{n+p-1}, x_{n+p})$$

$$= \rho(T^n x_0, T^n x_1) + \rho(T^{n+1} x_0, T^{n+1} x_1) + \cdots + \rho(T^{n+p-1} x_0, T^{n+p-1} x_1).$$

Thus by (1)

$$\rho(x_n, x_m) \leq \alpha^n \rho(x_0, x_1) + \alpha^{n+1} \rho(x_0, x_1) + \cdots + \alpha^{n+p-1} \rho(x_0, x_1)$$

$$\leq \alpha^n \rho(x_0, x_1)(1 + \alpha + \alpha^2 + \cdots).$$

and hence

$$\rho(x_n, x_m) \leq \frac{\alpha^n \rho(x_0, x_1)}{1 - \alpha}.$$
Exercises 6.4

1. Prove that \( R_d \) is complete.
2. Prove that the interval \((0, 1)\) with absolute value metric is not a complete metric space. 
   Prove that \((0, 1)\) with the metric of \( R_d \) is a complete metric space.
3. Prove that \( R^2 \) is complete.
4. Prove that \( l^\infty \) is complete. (Model your proof after the proof that \( l^2 \) is complete.)
5. If
   \[ T(x) = x^2 \quad (0 < x < \frac{1}{2}) \]
   prove that \( T \) is a contraction on \((0, \frac{1}{2})\), but that \( T \) has no fixed point.
6. If \( T : [0, 1] \rightarrow [0, 1] \) and there is a real number \( \alpha \) with \( 0 < \alpha < 1 \) such that
   \[ |T'(x)| < \alpha \quad (0 < x < 1) \]
   where \( T' \) is the derivative of \( T \), prove that \( T \) is a contraction on \([0, 1]\).
7. Let \( M \) be a metric space that is both totally bounded and complete. Prove that every sequence of points of \( M \) has a subsequence that converges to a point of \( M \).
8. Let \( M = [0, \infty) \) with the absolute value metric \( \rho(x, y) = |x - y| \). Let
   \[ f(x) = \frac{1}{1 + x^2} \quad (0 < x < \infty) \]
   Show that \( f : M \rightarrow M \), that
   \[ \rho(f(x), f(y)) = \rho(x, y) \quad (x, y \in M) \]
   but that \( f \) has no fixed point.

6.5 COMPACT METRIC SPACES

It is because the closed bounded interval \([a, b]\) is compact that many of the theorems about continuous functions on \([a, b]\) hold. We now begin a general discussion of compact metric spaces.

6.5A. DEFINITION. The metric space \( \langle M, \rho \rangle \) is said to be compact if \( \langle M, \rho \rangle \) is both complete and totally bounded.

For example, the metric space \([a, b]\) (with absolute value metric) is totally bounded and, by 6.4C, is complete. Hence \([a, b]\) is compact. The space \( R^1 \) is complete but not totally bounded. Hence \( R^1 \) is not compact. The metric space \((0, 1)\) (with absolute value metric) is totally bounded but not complete and hence is not compact.

From 6.3E we see that an infinite subset of \( R_d \) cannot be compact. We leave it to the reader to show that every finite subset of \( R_d \) actually is compact. (See Exercise 2.)

A very useful reformulation of compactness is continued in the following.

6.5B. THEOREM. The metric space \( \langle M, \rho \rangle \) is compact if and only if every sequence of points in \( M \) has a subsequence converging to a point in \( M \).

PROOF: Suppose first that \( M \) is compact and that \( \{ x_n \}_{n=1}^\infty \) is any sequence of points in \( M \). Then, by 6.3H, since \( M \) is totally bounded, the sequence \( \{ x_n \}_{n=1}^\infty \) has a Cauchy subsequence \( \{ x_{n_k} \}_{k=1}^\infty \). But \( \{ x_{n_k} \}_{k=1}^\infty \) converges to a point in \( M \) since \( M \) is complete. Thus if \( M \) is compact, then every sequence in \( M \) contains a convergent subsequence.

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10.3 PICARD EXISTENCE THEOREM FOR DIFFERENTIAL EQUATIONS

That is if probable," while Σ" refers to the k for which precisely k heads in n tosses is "less probable."

Indeed, the proof of 10.2A that we have given is essentially the same as one of the more familiar proofs of the "weak law of large numbers."

10.2F. We have proved that the set of all polynomials is dense in C[0, 1]. It is natural to ask whether we actually need all polynomials.

Specifically, let \( N = \{n_i\}_{i=1}^{\infty} \) be a strictly increasing sequence of positive integers, and let \( \mathcal{P}_N \) be the set of all polynomials \( P \) of the form

\[
P(x) = a_0 + a_1 x^{n_1} + a_2 x^{n_2} + \cdots + a_k x^{n_k}.
\]

That is, \( P \in \mathcal{P}_N \) if \( P \) is a constant plus a polynomial whose exponents all belong to \( N \). Here is a striking result (whose proof is beyond the scope of this book).

THE MUNTZ-SZASZ THEOREM. The set \( \mathcal{P}_N \) is dense in \( C[0, 1] \) if and only if

\[
\sum_{i=1}^{\infty} \frac{1}{n_i} = \infty.
\]

Thus for example, the set of all polynomials of the form

\[
a_0 + a_1 x^2 + a_2 x^4 + a_3 x^8 + \cdots + a_k x^{2^k}
\]

is not dense in \( C[0, 1] \) since here, \( n_i = 2^i \) so that \( \Sigma_{i=1}^{\infty} \frac{1}{n_i} = \Sigma_{i=1}^{\infty} \frac{1}{2^i} < \infty \). On the other hand, the set of all polynomials with even exponents is dense in \( C[0, 1] \).

Exercises 10.2

1. Calculate \( B_1, B_2, \) and \( B_3 \) for \( f \) where

\[
f(x) = x^2 \quad (0 < x < 1).
\]

Then graph these functions.

10.3 PICARD EXISTENCE THEOREM FOR DIFFERENTIAL EQUATIONS

Many problems in a course in elementary differential equations involve the solution of equations of the form

\[
\frac{dy}{dx} = f(x, y)
\]

with initial condition

\[
y(x_0) = y_0.
\]

Here \( f \) is, of course, some real-valued function defined on all or part of \( R^2 \). By a solution we mean a function \( \varphi \) with domain containing some interval \( [x_0 - \delta, x_0 + \delta] \) such that \( \varphi(x_0) = y_0 \) and

\[
\varphi'(x) = f[x, \varphi(x)] \quad (|x - x_0| < \delta).
\]

This is equivalent (via integration) to the equation

\[
\varphi(x) = y_0 + \int_{x_0}^{x} f[t, \varphi(t)] \, dt \quad (|x - x_0| < \delta).
\]
Thus the question of the existence of a solution to the problem posed by (1) and (2) is equivalent to the existence of a function \( \varphi \) satisfying (4) for some \( \delta \). We now prove a theorem, due to Picard, which gives conditions on \( f \) sufficient to ensure both the existence and the uniqueness of a function \( \varphi \) satisfying (4).

**Theorem.** If \( f \) is continuous on some rectangle \( D \subset \mathbb{R}^2 \) whose interior contains \( \langle x_0, y_0 \rangle \), and if there exists \( M > 0 \) such that 
\[
|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2| \quad (\langle x, y_1 \rangle, \langle x, y_2 \rangle \in D),
\]
then there exists \( \delta > 0 \) and a unique function \( \varphi \) such that (4) holds.

**Proof:** Since \( f \) is continuous on the compact set \( D \), we know (by 6.6B) that there exists \( k > 0 \) such that 
\[
|f(x,y)| \leq k \quad (\langle x, y \rangle \in D).
\]
Choose \( \delta > 0 \) such that
\[
\langle x, y \rangle \in D \quad \text{if} \quad |x - x_0| < \delta, |y - y_0| < k\delta
\]
and such that
\[
M\delta < 1
\]
where \( M \) is as in (5).

Let \( C^* \) be the subset of \( C[x_0 - \delta, x_0 + \delta] \) consisting of all functions \( \varphi \) that are continuous on \( [x_0 - \delta, x_0 + \delta] \) and such that
\[
|\varphi(x) - y_0| < k\delta, \quad (|x - x_0| < \delta).
\]

Then, by 10.1F, \( C^* \) is a complete metric space. Note that by (6) we have \( \langle t, \varphi(t) \rangle \in D \) if \( |t - x_0| < \delta \) and \( \varphi \in C^* \).

We now define a function \( T \) on \( C^* \) as follows: For \( \varphi \in C^* \) define \( T\varphi = \psi \) as
\[
\psi(x) = y_0 + \int_{x_0}^{x} f[t, \varphi(t)] \, dt \quad (|x - x_0| < \delta).
\]

Then \( \varphi \) satisfies (4) if and only if \( T\varphi = \varphi \).

We will show that \( T \) is a contraction (6.4E) on the complete metric space \( C^* \). First we show that \( T \) maps \( C^* \) into \( C^* \). Indeed, if \( \varphi \in C^* \) and \( \psi = T\varphi \), it is easy to show that \( \psi \) is continuous on \( [x_0 - \delta, x_0 + \delta] \). Moreover, if \( |x - x_0| < \delta \), then
\[
|\psi(x) - y_0| = \int_{x_0}^{x} |f[t, \varphi(t)]| \, dt < k|x - x_0| < k\delta.
\]
Hence, \( |\psi(x) - y_0| < k\delta \) if \( |x - x_0| < \delta \), and so \( \psi \in C^* \). Hence, \( T: C^* \to C^* \).

To show that \( T \) is a contraction suppose \( \varphi_1, \varphi_2 \in C^* \) and let \( \psi_1 = T\varphi_1, \psi_2 = T\varphi_2 \). Then, from (8), if \( |x - x_0| < \delta \), we have
\[
\psi_1(x) - \psi_2(x) = \int_{x_0}^{x} \{ f[t, \varphi_1(t)] - f[t, \varphi_2(t)] \} \, dt,
\]
and so
\[
|\psi_1(x) - \psi_2(x)| \leq \int_{x_0}^{x} |f[t, \varphi_1(t)] - f[t, \varphi_2(t)]| \, dt.
\]

* To be precise, say \( D = \{ \langle x, y \rangle \mid |x - x_0| < a, |y - y_0| < b \} \) for some \( a > 0, b > 0 \).

† The condition (5) is called a Lipschitz condition.
Using (5) we then obtain
\begin{equation}
|\psi_1(x) - \psi_2(x)| \leq M\int_{x_0}^{x} |\varphi_1(t) - \varphi_2(t)| \, dt
\end{equation}
\begin{equation}
< M\int_{x_0}^{x} \|\varphi_1 - \varphi_2\| \, dt
\leq M\delta \|\varphi_1 - \varphi_2\|.
\end{equation}
Thus
\begin{equation}
\|\psi_1 - \psi_2\| < M\delta \|\varphi_1 - \varphi_2\|.
\end{equation}
or
\begin{equation}
\|T\varphi_1 - T\varphi_2\| < M\delta \|\varphi_1 - \varphi_2\|.
\end{equation}
With respect to the metric \(\rho\) for \(C^*\) this reads
\begin{equation}
\rho(T\varphi_1, T\varphi_2) < M\delta \rho(\varphi_1, \varphi_2).
\end{equation}
In view of (7) this proves that \(T\) is a contraction on \(C^*\). Hence, by 6.4F, there is precisely one \(\varphi \in C^*\) such that \(T\varphi = \varphi\). But the definition (8) of \(T\) shows that \(T\varphi = \varphi\) means that (4) holds. This completes the proof.

Our proof uses both the continuity of \(f\) and the Lipschitz condition (5) to show that a solution
does not exist and is unique.

It is possible, using a different method of proof, to show that a solution exists assuming only the continuity of \(f\) but not the Lipschitz condition. See Section 10.6.

However, the Lipschitz condition is necessary in order to prove that the solution is unique. For example, both \(\varphi_1(x) = 0\) and \(\varphi_2(x) = x^2/27\) are solutions to
\begin{equation}
\frac{dy}{dx} = y^{2/3}, \quad y(0) = 0.
\end{equation}
There is thus no unique solution to (9). We leave it to the reader to show that \(f(x, y) = y^{2/3}\) does not satisfy a Lipschitz condition in any rectangle \(D\) about \((0, 0)\).

(Show that
\begin{equation}
|f(0, y) - f(0, 0)| < My \quad (0, y) \in D
\end{equation}
holds for no \(M\).)

**Exercises 10.3**

1. Show that \(\varphi\) is a solution to
\begin{equation}
\frac{dy}{dx} = x + y, \quad y(0) = 0
\end{equation}
if and only if
\begin{equation}
\varphi(x) = \int_{0}^{x} [t + \varphi(i)] \, dt.
\end{equation}
Define \(T\) as follows: For any \(\varphi\) let \(T\varphi = \psi\) where
\begin{equation}
\psi(x) = \int_{0}^{x} [t + \varphi(i)] \, dt.
\end{equation}
Let $\varphi_0 = 0$. If $\varphi_1 = T \varphi_0$, show that $\varphi_1(x) = x^2/2!$. If $\varphi_2 = T \varphi_1$, show that $\varphi_2(x) = x^3/2! + x^3/3!$. In general, if $\varphi_n = T \varphi_{n-1}$, show that $\varphi_n(x) = x^2/2! + x^3/3! + \cdots + x^n/n!$. Then show that
\[
\lim_{n \to \infty} \varphi_n(x) = \lim_{n \to \infty} T^n \varphi_0(x) = e^x - x - 1.
\]
Verify that $\varphi(x) = e^x - x - 1$ is a solution to the original problem.

Compare this method of solution with the proof of 6.4F.

2. Use the same method to solve
\[
y' = y, \quad y(1) = 1.
\]

3. Use the same method to solve
\[
y' = x - y, \quad y(0) = 1.
\]

### 10.4 The Arzela Theorem on Equicontinuous Families

10.4A. In higher analysis it is often useful to know when a sequence of continuous functions on $[a,b]$ will have a uniformly convergent subsequence.

For an example where this does not happen consider $(f_n)_{n=1}^\infty$ where
\[
f_n(x) = x^n \quad (0 < x < 1).
\]

As we have seen, $(f_n)_{n=1}^\infty$ converges pointwise on $[0,1]$ to the discontinuous function $f$ where
\[
f(x) = 0 \quad (0 < x < 1),
f(1) = 1.
\]

Any subsequence of $(f_n)_{n=1}^\infty$ must, therefore, also converge pointwise to $f$. Hence, by 9.3C, no subsequence of $(f_n)_{n=1}^\infty$ can converge uniformly on $[0,1]$, since $f$ is not continuous.

The condition we are seeking involves the concept of equicontinuity.

10.4B. Definition. Let $[a,b]$ be a closed bounded interval. The subset $\mathcal{F}$ of $C[a,b]$ is said to be equicontinuous if given $\epsilon > 0$, there exists $\delta > 0$ such that
\[
|f(x) - f(y)| < \epsilon \quad (|x - y| < \delta, f \in \mathcal{F}).
\]

That is, $\mathcal{F}$ is an equicontinuous subset of $C[a,b]$ if, given $\epsilon > 0$, there exists $\delta$ independent of $f$ such that $|f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$. The same $\delta$ must work for all $f \in \mathcal{F}$ as well as for all $x, y$.

A condition sufficient to ensure that a sequence of continuous functions on $[a,b]$ has a uniformly convergent subsequence will come out of the next result (which is known as Arzela's theorem or as Ascoli's theorem).

10.4C. Theorem. Let $\mathcal{F}$ be a bounded equicontinuous subset of the metric space $C[a,b]$. Then $\mathcal{F}$ is totally bounded.

**Proof:** Given $\epsilon > 0$ we must show that there are a finite number of subsets $A_i$ of $C[a,b]$ such that diam $A_i < \epsilon$ and $\mathcal{F} \subset A_i$.

Since $\mathcal{F}$ is bounded there exists $M > 0$ such that
\[
\rho(f,0) = \|f\| < M \quad (f \in \mathcal{F}).
\]

Since $\mathcal{F}$

1. **Subdivision**
   - $\cdots < \cdots < b, - \lambda$
   - $\epsilon / g(x_i)$

This is $\mathcal{F}$ consists...

and so,

Since $tl$

Now $f_c$

Thus $b$

Thus $f_i$

The of there $c$

The $n$-numbe... finite $\mathcal{F}$ contai

Her

10.4D. Theorem. The $f_n$ form a convergent...