An explicit example of Hopf bifurcation in fluid mechanics

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It is observed that a complete and explicit example of Hopf bifurcation appears not to be known in fluid mechanics. Such an example is presented for the rotating Bénard problem with free boundary conditions on the upper and lower faces, and horizontally periodic solutions. Normal modes are found for the linearization, and the Veronis computation of the wave numbers is modified to take into account the imposed horizontal periodicity. An invariant subspace of the phase space is found in which the hypotheses of the Joseph–Sattenger theorem are verified, thus demonstrating the Hopf bifurcation. The criticality calculations are carried through to demonstrate rigorously, that the bifurcation is subcritical for certain cases, and to demonstrate numerically that it is subcritical for all the cases in the paper.

1. Introduction

Transitions from a stable steady state to a stable time-periodic state as an appropriate parameter is varied are well attested in many physical systems. For systems modelled by ordinary differential equations, the van der Pol and the Zhbottinski–Belousov equations for example (Marsden & McCracken 1976), such transitions have not only been observed experimentally but have also been proven mathematically to occur in the model equations. For systems modelled by partial differential equations, particularly in hydrodynamics, the situation is not generally as completely known. Certainly, in classical fluid mechanical experiments on convection, such as the Bénard experiment of a layer of fluid heated from below and its variations, transitions from a steady conducting or convective state to a time-periodic convective state are well known. Furthermore, it is well known that such transitions are associated with the occurrence of overstability (Chandrashekar 1981) in the linearization of the modelling equations about the steady state, that is when the spectrum lies entirely in the left half of the complex plane, except for a strictly complex conjugate pair of eigenvalues which crosses the imaginary axis as a parameter (usually the Rayleigh number) is varied.

For finite-dimensional evolutionary systems of ordinary differential equations, a celebrated theorem of E. Hopf (Marsden & McCracken 1976) states that a periodic
solution occurs from an overstable transition if in addition the complex conjugate pair of eigenvalues has simple multiplicity and satisfy a mild transversality condition. The transition is then called a Hopf bifurcation. It may not however result in a stable periodic solution. If the steady state loses stability as the bifurcation parameter increases past a certain critical value, then the Hopf theorem says that if periodic solutions appear for parameter values greater than the critical value then they are stable, whereas if they appear for smaller values then they are unstable. The former supercritical case, unlike the latter subcritical case, is the physically meaningful case as only stable solutions can actually be observed. The Hopf theorem has been extended by Iooss (1972), Joseph & Sattinger (1972) and others to evolutionary systems of partial differential equations, in particular to the Navier–Stokes equations. There are consequently manageable criteria for detecting Hopf bifurcations in a wide range of systems, although the problem of determining the criticality of the bifurcation remains. Marsden & McCracken (1976) have developed an explicit algorithm for determining criticality in a finite-dimensional system, but for partial differential equations, especially those in a hydrodynamics, a more practicable algorithm is based on a finite amplitude power series method of Veronis (1959), which was used, independently, by Joseph & Sattinger (1972) in their proof of the Hopf theorem for the Navier–Stokes equations.

In his 1975 review of bifurcation theory in hydrodynamics, Kirchgässner commented that not a single explicit example of a Hopf bifurcation in fluid mechanics appeared known (Kirchgässner 1975). Since then, Iooss et al. (1978) have given an example involving the Ekman boundary layer equations, but this example is incomplete as the simplicity of the complex conjugate pair of eigenvalues is conjectured and the criticality of the bifurcation determined numerically. In this paper we will show that the steady conducting state in the rotating Bénard problem loses stability through a subcritical Hopf bifurcation; accordingly a finite amplitude change of state would be observed provided the Taylor number $\mathcal{T}^2$ (the rotation rate) is sufficiently high and the Prandtl number $\sigma$ (the ratio of viscosity to thermal diffusivity) sufficiently small. For this we show algebraically that all of the conditions on the spectrum, including simplicity of the complex conjugate pair of eigenvalues that cross the imaginary axis, are satisfied and also determine algebraically a formula in $\mathcal{T}^2$ and $\sigma$, the sign of which indicates the criticality of the bifurcation. Our example is thus both complete and explicit.

The principal motivation for finding such an example is that overstability alone does not suffice to produce a Hopf bifurcation, and thus a branching periodic solution. A priori one may expect that overstability with multiplicity 2 leads to phase locked periodic solutions, but with multiplicity greater than 2 one must take into account the possibility that the solution may bifurcate to a strange attractor (Ruelle & Takens 1971a,b). In our example, this possibility actually remains because our calculation shows that the bifurcation is subcritical. Consequently, the periodic is unstable and the steady solution loses stability to some solution which is not a Hopf periodic solution.
Our calculations begin in §2, in which we set up the rotating Bénard problem. The boundary conditions (free boundary conditions even on vertical boundaries) are chosen to make tractable the computations of §§6 and 7 which determine criticality. Since these boundary conditions allow extraneous rigidly rotating solutions, we introduce an averaging condition (2.6) which eliminates extraneous solutions. The resulting problem has even multiplicity in all eigenvalues so that crucial condition of multiplicity 1 cannot be realized. Accordingly, we find an invariant subspace by imposing parity conditions (2.7) and restrict to it. Now it is possible to carry out the calculation, which begins with the linearization of the problem about the canonical rigidly rotating conducting solution. The linear problem decomposes into five, three and one dimensional problems, which make it possible to find the eigenvalues and eigenfunctions. We recover the formulae of Veronis (1959) relating the eigenvalues to the heating, Prandtl number, rotation rate and wave number, and from there on follow his analysis to find the virtual wave numbers minimizing the heating at which critical and overstable bifurcations take place. Since the geometric constraints do not allow every virtual wave number to be realized as an actual wave number, we then alter his analysis to take this fact into account. Thus we obtain formulae like those of Veronis (1959) relating the critical and overstable heatings to the Prandtl number, rotation rate and corresponding wave numbers and eigenvalues.

In §3 we use these formulae to find conditions on the geometry of the problem and on the Prandtl number and the rotation rate under which both the overstable heating is less than the critical heating and the imaginary eigenvalue occurs with multiplicity 1.

In §4, we recall the work of Joseph & Sattinger (1972) to show that the hypothesis of their theorem is satisfied, so that a Hopf bifurcation will take place. In §5 we adapt their criterion for criticality to our situation to obtain an algorithm.

Up to this point, the free boundary conditions have not been necessary, but in §6 we seek to apply the Joseph–Sattinger algorithm to an explicit situation, and it is to keep the calculations in this section within reasonable bounds that we adopt that boundary condition.

In §7 we evaluate the formula obtained in §6 for a range of values of the Prandtl number, which the rotation rate depending on them, to conclude that for this range the Hopf bifurcations are all subcritical.

2. THE BÉNARD PROBLEM AND ITS LINEARIZATION

In the Bénard problem, one studies the onset of convection in an infinite horizontal slab of viscous fluid which is rotating rigidly and which is heated uniformly from below. We restrict our attention to the function space of possible solutions which have some fixed horizontal periodicity, and further we impose free boundary conditions at the top and bottom boundaries. As usual, we take $x, y, z, t$ to be the rotating space and time coordinates, $u, v, w$ to be the three components of velocity, $\theta$ to be the
temperature and \( P \) to be the pressure. Thus, we suppose at the outset that the following two conditions are satisfied; we are given \( \ell_1 > 0 \) and \( \ell > 0 \).

**Periodicity conditions**

For \( f = u, v, w, \theta \) or \( P \),

\[
\begin{align*}
\frac{f(x + 2\ell_1, y, z, t)}{f(x, y, z, t),} \\
\frac{f(x, y + 2\ell_2, z, t)}{f(x, y, z, t).}
\end{align*}
\]

(2.1)

The top and bottom boundaries of the slab we may take to be at \( z = 1 \) and \( z = 0 \) respectively.

**Boundary conditions, I**

\[
0 = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z}, \quad \text{at} \quad z = 0, 1, \quad (2.2)
\]

\[
\theta = T_0 \quad \text{at} \quad z = 0, \quad \theta = T_1 \quad \text{at} \quad z = 1. \quad (2.3)
\]

Recall that the Boussinesq approximation to the Navier–Stokes equations will have solutions in which \( u, v, w \) describe rigid rotation and in which \( \theta \) depends linearly on \( z \). Choose a non-zero angular velocity and its corresponding solution, then expand the Navier–Stokes equations about the solution. We obtain for the dimensionless equations of motion,

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\frac{1}{\sigma} \frac{\partial P}{\partial x} + \sigma \Delta u - \sigma \mathcal{F} v - \mathbf{v} \cdot \nabla u, \\
\frac{\partial v}{\partial t} &= -\frac{1}{\sigma} \frac{\partial P}{\partial y} + \sigma \Delta v + \sigma \mathcal{F} u - \mathbf{v} \cdot \nabla v, \\
\frac{\partial w}{\partial t} &= -\frac{1}{\sigma} \frac{\partial P}{\partial z} + \sigma \Delta w + \theta - \mathbf{v} \cdot \nabla w, \\
\frac{\partial \theta}{\partial t} &= \Delta \theta + \sigma \lambda w - \mathbf{v} \cdot \nabla \theta, \\
0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},
\end{align*}
\]

(2.4)

where now \((u, v, w) = \mathbf{v}\) describes the departure of the velocity from the given rotational rigid motion; \(\theta\) is the departure of the temperature from the linear rigid solution temperature. The parameter \(\mathcal{F}\) describes the given rate of rotation, \(\sigma\) is the Prandtl number and \(\lambda\) is the heating, proportional to \(T_0 - T_1\). The boundary conditions satisfied by \(u, v, w, \theta\) are:

**boundary conditions, II**

\[
0 = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \theta. \quad (2.5)
\]

Now, \(0 = u = v = w = \theta\) is a solution to the problem set by (2.1), (2.4) and (2.5) any \(\lambda\), but unfortunately it is not unique for any value of \(\lambda\); thus the usual stability analysis will not apply. The reason the solution is not unique is that the rigid solution
corresponding to any other rate of rigid rotation determines a solution of (2.1), (2.4) and (2.5). To exclude these solutions we impose the condition that the average horizontal velocities vanish.

**Average velocity condition**

\[
0 = \int_0^{2\ell_1} \int_0^{2t} u(x,y,z) \, dx \, dy = \int_0^{2\ell_1} \int_0^{2t} v(x,y,z) \, dx \, dy.
\]  

(2.6)

Now, the trivial solution is unique and asymptotically stable for small \(\lambda\).

At this point we have a reasonable function space in which to seek solutions of the Bénard problem with free boundary conditions. However, as it will turn out, the hypotheses of the Hopf Bifurcation theorems of Joseph–Sattinger and Iooss cannot be satisfied in this function space: the required simplicity of the imaginary eigenvalue never occurs. Since, in this paper, we seek a true Hopf bifurcation, we restrict our function space yet again to exclude the undesirable eigenfunctions. We do so by imposing a parity condition of the sort imposed to obtain steady bifurcation in Kirchgässner (1975).

**Parity condition**

\[
\begin{align*}
u(x,y,z,t) &= -u(-x,-y,z,t), \\
v(x,y,z,t) &= -v(-x,-y,z,t), \\
w(x,y,z,t) &= w(-x,-y,z,t), \\
\theta(x,y,z,t) &= \theta(-x,-y,z,t), \\
P(x,y,z,t) &= P(-x,-y,z,t).
\end{align*}
\]  

(2.7)

It is easy to check that (2.7) describes a subspace of our function space, invariant under the Navier–Stokes semi-flow, so that it makes sense to seek Hopf bifurcations in this subspace. Also, it is now easy to see that our function space \(\mathcal{H}\) consists of a suitable completion of the space of finite sums of vector functions of the form

\[
\mathcal{V} = \begin{pmatrix}
U \sin \left[ \frac{\pi a_1}{\ell_1} x + \frac{\pi a_2}{\ell_2} y \right] \cos \left( \pi nz \right) \\
V \sin \left[ \frac{\pi a_1}{\ell_1} x + \frac{\pi a_2}{\ell_2} y \right] \cos \left( \pi nz \right) \\
W \cos \left[ \frac{\pi a_1}{\ell_1} x + \frac{\pi a_2}{\ell_2} y \right] \sin \left( \pi nz \right) \\
T \cos \left[ \frac{\pi a_1}{\ell_1} x + \frac{\pi a_2}{\ell_2} y \right] \cos \left( \pi nz \right) \\
P \cos \left[ \frac{\pi a_1}{\ell_1} x + \frac{\pi a_2}{\ell_2} y \right] \cos \left( \pi nz \right)
\end{pmatrix},
\]  

(2.8)

with \(a_1, a_2, n\) integral, \(n \geq 0\), \(a_1 \geq 0\) and \(a_2 \geq 0\) if \(a_1 = 0\), and with \(U, V, W, T, P\) real. For fixed \(a_i/\ell_i\) and \(n\), we denote by \(\mathcal{H}(\alpha, n)\) the vector space consisting of vector functions of the form (2.3). This space is five dimensional when \(\alpha \neq 0\) and \(n \neq 0\); otherwise it has lower dimension. Then \(\mathcal{H}\) as a Hilbert space is the orthogonal direct sum of the subspaces \(\mathcal{H}(\alpha, n)\), with norm in \(\mathcal{H}(\alpha, n)\) given by

\[
\| \mathcal{V} \|^2 = \zeta(\alpha, n) \left[ U^2 + V^2 + W^2 + T^2 + P^2 \right],
\]  

(2.9)

when \(\alpha \neq 0\) and \(n \neq 0\), with similiar formulæ when \(\alpha = 0\) or \(n = 0\). The choice of the positive numbers \(\zeta(\alpha, n) > 0\) determines which completion \(\mathcal{H}\) is, but it makes no
difference to the spectral analysis of the linearization of the Navier–Stokes equations about the rigid solution chosen above.

The linearization about the rigid solution is the system of partial differential equations given by deleting the nonlinear terms in (2.4). Specifically, it is the evolutionary system in $\mathbb{H}$ given by

$$
\begin{pmatrix}
\frac{\partial u}{\partial t} \\
\frac{\partial v}{\partial t} \\
\frac{\partial w}{\partial t} \\
\frac{\partial \theta}{\partial t} \\
0
\end{pmatrix} = \mathcal{L}
\begin{pmatrix}
u \\
v \\
w \\
\theta \\
P
\end{pmatrix},
$$

(2.10)

where $\mathcal{L}$ is defined by

$$
\begin{pmatrix}
\frac{\partial P}{\partial x} + \sigma \Delta u + \sigma \mathcal{F} v \\
\frac{\partial P}{\partial y} + \sigma \Delta v + \sigma \mathcal{F} u \\
\frac{\partial P}{\partial z} + \sigma \Delta w + \theta \\
\Delta \theta + \sigma \lambda w \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\end{pmatrix}.
$$

(2.11)

The subspace $\mathbb{H}(\alpha, n)$ is invariant under $\mathcal{L}$, and with respect to the natural basis in $\mathbb{H}(\alpha, n)$, the system (2.10) induces a finite dimensional linear system in $\mathbb{H}(\alpha, n)$. When $\alpha \neq 0$ and $n \neq 0$, this system is given by

$$
\begin{pmatrix}
\dot{U} \\
\dot{V} \\
\dot{W} \\
\dot{T} \\
0
\end{pmatrix} =
\begin{pmatrix}
-\pi^2(\alpha^2 + n^2) \sigma & -\sigma \mathcal{F} & 0 & 0 & \pi \alpha_1 / \sigma \\
\sigma \mathcal{F} & -\pi^2(\alpha^2 + n^2) \sigma & 0 & 0 & \pi \alpha_2 / \sigma \\
0 & 0 & -\pi^2(\alpha^2 + n^2) & 1 & \pi n \\
0 & 0 & \sigma \lambda & -\pi^2(\alpha^2 + n^2) & 0 \\
\pi \alpha_1 & \pi \alpha_2 & \pi n & 0 & 0
\end{pmatrix}
\begin{pmatrix}
U \\
V \\
W \\
T \\
P
\end{pmatrix},
$$

(2.12)

with similar systems induced when $\alpha = 0$ or $n = 0$. Again with $\alpha \neq 0, n \neq 0$, write $N$ for the subspace of $\mathbb{R}^4$ consisting of four-vectors with components $U, V, W$ and $T$, satisfying

$$
\pi \alpha_1 U + \pi \alpha_2 V + \pi n W = 0.
$$

(2.13)

Let $Q$ be the matrix of orthogonal projection of $\mathbb{R}^4$ on $N$. Writing the matrix of (2.12) in block form

$$
\begin{pmatrix}
A & (1/\sigma) b^T \\
b & 0
\end{pmatrix},
$$
we see that the system (2.12) is equivalent to the three dimensional system
\[ \dot{\zeta} = QA\zeta, \zeta \in \mathbb{N}. \] (2.14)

The characteristic roots of the system (2.14) (that is, the eigenvalues of $QAQ$) are exactly the roots of the polynomial in $p$,
\[ p^3 + (2\sigma + 1)\pi^2(\alpha^2 + n^2)p^2 + \pi^4(\alpha^2 + n^2)^2(\sigma^2 + 2\sigma) 
+ (T^2n^2\sigma - \alpha^2\lambda)\sigma/(\alpha^2 + n^2)\] 
\[ + \sigma^2\pi^6(\alpha^2 + n^2)^3 + \sigma^2\pi^2(T^2n^2 - \alpha^2\lambda) = 0, \] (2.15)

which is the same polynomial as that obtained by Veronis (1959).

We have to dispose of the anomalous cases of $(\alpha, n)$ before turning to the spectrum of (2.10). If $\alpha = 0$ and $n = 0$, $H(\alpha, n)$ is one dimensional and $N$ is zero dimensional. That is, this case contributes nothing. If $\alpha \neq 0$ while $n = 0$, $H(\alpha, n)$ is three dimensional and $N$ is one dimensional. Then (2.14) has a single characteristic root $-\sigma\pi^2\alpha^2$. If $\alpha = 0$ while $n \neq 0$, we obtain the same dimensions, but the single characteristic root is $-\sigma\pi^2n^2$.

The spectrum of the system (2.10) consists of $-\sigma\pi^2\alpha^2$, $-\sigma\pi^2n^2$, and the roots $p$ of (2.15), each counted once for each $H(\alpha, n)$ that produces it.

For $\lambda$ sufficiently negative, it is easy to check that all of these characteristic roots have negative real part. For $\lambda$ sufficiently large, at least one acquires a positive real part; always at most finitely many have non-negative real parts. We wish to find the smallest value of $\lambda$ at which some such real part vanishes. This is the value of $\lambda$ at which the rigid or conducting solution loses stability to some other such solution (Chandrashekar 1981).

We begin by considering the case in which the imaginary part also vanishes. The roots $-\sigma\pi^2\alpha^2$ and $-\sigma\pi^2n^2$ associated with $H(\alpha, 0)$ and $H(0, n)$ respectively vanish for no choice of $\lambda$. However, if $\alpha \neq 0$ and $n \neq 0$, setting $p = 0$ shows that the value of $\lambda$, for which a root of (2.15) vanishes, is given by
\[ \lambda_c(\alpha, n) = \pi^4(\alpha^2 + n^2)^3 + T^2n^2/\alpha^2. \] (2.16)

We ask for $\lambda_c(T^2)$, the smallest value of $\lambda_c(\alpha, n)$ as $(\alpha, n)$ ranges over all admissible values of $\alpha$ and $n$. Clearly we may assume $n = 1$. The right side of (2.16) takes on an absolute minimum at $\alpha^2 = \alpha_c^2$ the unique root of
\[ 2\alpha_c^6 + 3\alpha_c^4 - (1 + T^2/4\pi) = 0. \] (2.17)

Since $\alpha$ must have the form $\alpha = a_1/\ell_1, a_2/\ell_2$ with $a_i$ integral, we cannot simply assume $\alpha^2 = \alpha_c^2$. If $a_1$ and $a_2$ may be chosen integral so that $\alpha^2 = \alpha_c^2$, then we obtain Veronis’s result that
\[ \lambda(T^2) = 3\pi^4(\alpha_c^2 + 1)^2. \] (2.18)

However, in general we need to observe that the elements of the set
\[ \{\alpha^2|\alpha^2 = a_1^2/\ell_1^2 + a_2^2/\ell_2^2, a_i \text{ integral}\} \]
may be listed as a sequence \(0 = A_1 < A_1 < A_2 < \ldots\) with \(\lim_{n \to \infty} A_n = \infty\). Since \(A_n\) diverges to \(\infty\) and since \(\lambda_c(x, 1)\) has a unique minimum, the number \(\lambda_c(\mathcal{T}^2)\) we seek is well defined by

\[
\lambda_c(\mathcal{T}^2) = \min \{[\pi^4(A_i + 1)^3 + \mathcal{T}^2]/A_i | i = 1, 2, 3, \ldots\}. \tag{2.19}
\]

Suppose that \(j\) is the index for which the quantity being minimized attains its minimum and suppose that that quantity attains the same minimum value at the index \(i\). Since \([\pi^4(A^2 + 1)^3 + \mathcal{T}^2]/A^2\) has a unique minimum, it follows that \(|i - j| = 1\). Using this fact, we may describe the function \(\lambda_c(\mathcal{T}^2)\) more precisely. First, define

\[
\mathcal{T}^2_i = \pi^4[A_{i+1}A_i(A_{i+1} + A_i) + 3A_{i+1}A_i - 1], \tag{2.20}
\]

with the proviso that the left side is undefined whenever the right side is negative. Thus we obtain an increasing sequence \(0 < \mathcal{T}^2_k < \mathcal{T}^2_{k+1} < \mathcal{T}^2_{k+2} < \ldots\) and then it is easy to see that

\[
\lambda_c(\mathcal{T}^2) = [\pi^4(A_i + 1)^3 + \mathcal{T}^2]/A_i \quad \text{for} \quad \mathcal{T}^2_{i-1} < \mathcal{T}^2 < \mathcal{T}^2_i, \tag{2.21}
\]

where we extend (2.20) by defining \(\mathcal{T}^2_{k-1} = 0\). Notice that \(\lambda_c(\mathcal{T}^2)\) is an increasing, piecewise linear, continuous function of \(\mathcal{T}^2\), with decreasing slope and

\[
\lim_{\mathcal{T}^2 \to \infty} \lambda_c(\mathcal{T}^2) = \infty. \tag{2.22}
\]

Finally, we can conclude that for \(\lambda < \lambda_c(\mathcal{T}^2)\) all the real characteristic roots of the system (2.10) are negative, no other roots are zero, and further we conclude that for \(\lambda = \lambda_c(\mathcal{T}^2)\), the root zero appears with multiplicity at least the number of ways that \(A_i\) may be written as \(a_i^2/f_i^2 + a_3^2/f_3^2\) with \(a_i\) integral and \(a_1 > 0\) or \(a_1 = 0\) and \(a_2 > 0\).

Now we turn to \(\lambda_H(\mathcal{T}^2, \sigma)\), the smallest value of the Rayleigh number \(\lambda\) for which (2.10) has a purely imaginary root. Following Veronis (1959), we see that (2.15) will have a purely imaginary root if and only if \(\lambda = \lambda_H(x, n)\) and

\[
[(1 - \sigma)/(1 + \sigma)] \mathcal{T}^2n^2 > \pi^4(x^2 + n^2)^3, \tag{2.23}
\]

where

\[
\lambda_H(x, n) = 2(\sigma + 1)[\pi^4(x^2 + n^2)^3 + (\sigma^2/(\sigma + 1)^2) \mathcal{T}^2n^2]/x^2. \tag{2.24}
\]

Again, we seek the smallest \(\lambda_H(x, n)\) for given \(\sigma\) and \(\mathcal{T}^2\), subject this time to condition (2.23). Fortunately, the function \(f(n) = [(1 - \sigma)/(1 + \sigma)] \mathcal{T}^2n^2 - \pi^4(x^2 + n^2)^3\) is positive exactly on an interval of the form \((0, N)\) if it is ever positive. Consequently, if (2.23) holds for some \(n\), then it holds for \(n = 1\). For a fixed choice of \(\mathcal{T}^2\) and \(\sigma\), we always have

\[
\lambda_H(x, 1) \leq \lambda_H(x, n). \tag{2.25}
\]

Thus, for a given choice of \(\mathcal{T}^2\) and \(\sigma\), the smallest \(\lambda_H(x, n)\) subject to (2.23) will be \(\lambda_H(x, 1)\) subject to (2.23) with \(n = 1\). Now we may analyse \(\lambda_H(x, 1)\) as we did...
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\[ \lambda_e(x, 1), \text{ and again we observe that the curve } \lambda = \lambda_H(x, 1) \text{ has a unique minimum on } 0 < x < \infty \text{ at } x^2 = \alpha_H^2 \text{ where } \alpha_H^2 \text{ is the unique positive root of} \]

\[ \pi^4(2x^6 + 3x^4 - 1 - \sigma^2/(\sigma + 1)^2)\mathcal{T}^2 = 0. \]  

(2.25)

But again we require \( x^2 \) to have the form \( a_{i1}/l_1^2 + a_{i2}/l_2^2 \) with \( a_i \) integral. Arguing as before, with the same sequence, \( 0 = A_1 < A_2 < A_3 < \ldots \), we conclude that

\[ \lambda_H(\mathcal{T}^2, \sigma) = 2(\sigma + 1)\lambda_e[\sigma/(\sigma + 1)]\mathcal{T}^2. \]  

(2.26)

Consequently (2.21) computes \( \lambda_H \) as well,

\[ \lambda_H(\mathcal{T}^2, \sigma) = 2(\sigma + 1)\left[ \pi^4(A_i + 1)^2 + [\sigma^2/(\sigma + 1)]\mathcal{T}^2 \right]/A_i \]

for \( [(\sigma + 1)/(\sigma + 1)]\mathcal{T}^2 - 1 < \mathcal{T}^2 < [(\sigma + 1)/(\sigma + 1)]\mathcal{T}^2. \)  

(2.27)

We concluded that the dynamical system (2.10) in \( \mathbb{H} \) has no strictly imaginary roots for \( \lambda < \lambda_H(\mathcal{T}^2, \sigma) \), and that for \( \lambda = \lambda_H(\mathcal{T}^2, \sigma) \), provided that

\[ [(1 - \sigma)/(1 + \sigma)]\mathcal{T}^2 > \pi^4(A_i + 1)^2, \]  

(2.28)

the system has exactly two imaginary roots

\[ \pm P_1 = \pm (-1)^{1/2}\sigma[(\mathcal{T}^2(1 - \sigma)/(1 + \sigma) - \pi^4(A_i + 1)^2)/(A_i + 1)]^{1/2}. \]  

(2.29)

Furthermore, if \( [(\sigma + 1)/(\sigma + 1)]\mathcal{T}^2 - 1 < \mathcal{T}^2 < [(\sigma + 1)/(\sigma + 1)]\mathcal{T}^2 \), holds also, then the multiplicity of each of these roots is given by

\[ M_i = \text{number of } \{a_1, a_2 \mid a_1, a_2 \text{ integral}\}
\]

\[ \text{with } \quad a_{i1}/l_1^2 + a_{i2}/l_2^2 = A_i, \]

\[ \text{where, } \quad a_1 > 0 \quad \text{or} \quad a_1 = 0 \quad \text{and} \quad a_2 > 0 \]  

(2.30)

It is equally easy to calculate the multiplicity in case \( \mathcal{T}^2 = [(\sigma + 1)/(\sigma + 1)]\mathcal{T}^2 \), but we omit that calculation.

Finally, to apply the Hopf Bifurcation Theorem of Ioos (1972) and Joseph & Sattinger (1972), we will need to check one more condition: that the eigenvalues cross the imaginary axis with non-zero velocity as \( \lambda \) crosses \( \lambda_H(\mathcal{T}^2, \sigma) \). For the reader's convenience we recall briefly the calculation of Veronis (1959). Suppose that for \( \lambda = \lambda_H \), a pure imaginary eigenvalue occurs with multiplicity 1. Then for \( \lambda \) near \( \lambda_H \) we have two eigenvalues \( r(\lambda) \pm is(\lambda) \) depending smoothly on \( \lambda \), with \( r(\lambda_H) = 0 \) and \( is(\lambda_H) \) our pure imaginary eigenvalue. Then \( r(\lambda) \) satisfies a polynomial obtained easily from (2.15). Then implicit differentiation produces the equation

\[ 2[(2\sigma + 1)^2\pi^4(A_i + 1)^2 + \sigma^2(\mathcal{T}^2(1 - \sigma)/(1 + \sigma)) - \pi^4(A_i + 1)^2]/(A_i + 1)](dr/d\lambda)_{\lambda = \lambda_H} = A_i \pi^2(\sigma^2 + \sigma). \]  

(2.31)

Since the terms in the brackets and on the right hand side are all positive, we conclude that \( (dr/d\lambda)_{\lambda = \lambda_H} > 0 \).

We summarize these calculations in an easy proposition.
Proposition 2.2. Suppose that $\lambda_H(\mathcal{T}^2, \sigma) < \lambda_c(\mathcal{T}^2)$ and that in addition

$$[((\sigma + 1)/\sigma)^2 \mathcal{T}^2_{i-1} < \mathcal{T}^2 < [((\sigma + 1)/\sigma)^2 \mathcal{T}^2_i,$$

and

$$[((1 - \sigma)/(1 + \sigma))^2 \mathcal{T}^2 > \pi^4(A_i + 1)^3,$$

then the two characteristic roots $\pm P_1$ occur with multiplicity $M_i$ each when $\lambda = \lambda_H(\mathcal{T}^2, \sigma)$. Moreover, all other characteristic roots have negative real part for $\lambda = \lambda_H(\mathcal{T}^2, \sigma)$, as do all characteristic roots for $\lambda < \lambda_H(\mathcal{T}^2, \sigma)$. Finally, if $M_i = 1$, the corresponding complex roots for $\lambda$ near $\lambda_H(\mathcal{T}^2, \sigma)$ cross the imaginary axis with non-zero velocity as $\lambda$ passes $\lambda_H(\mathcal{T}^2, \sigma)$.

Of course, a more elaborate and less intelligible statement is easily obtained, for instance, to deal with the cases that $M_i > 1$ or $\mathcal{T}^2 = [((\sigma + 1)/\sigma)^2 \mathcal{T}^2_i$. We remark, however, that the condition $M_i = 1$ is crucial to our application. If $M_i > 1$, one expects an invariant torus of dimension $M_i$ to bifurcate at $\lambda = \lambda_H(\mathcal{T}^2, \sigma)$; not a periodic solution. For $M_i = 2$, one expects period locked and quasi-periodic solutions to alternate in a complex way (Ioss & Joseph 1980), assuming that one has stability at all. For $M_i \geq 3$, it is plausible that solenoids related to the Smale horseshoe (Smale 1967) will appear, or even more complicated invariant sets.

At this point we may detect a certain amount of stability.

Corollary. If $\lambda < \min(\lambda_c(\mathcal{T}^2), \lambda_H(\mathcal{T}^2, \lambda))$, the rigidly rotating solution is asymptotically stable.

Proof. The eigenvalues all have negative real part; then the result follows from the work of Kielhöfer & Kirchgässner (1973).

3. Verification of the hypotheses for Hopf bifurcation

In our model for rotating Bénard convection with free boundary conditions, we have four parameters at our disposal, $\sigma$, $\mathcal{T}^2$, $\ell_1$ and $\ell_2$. We wish to choose these in a reasonably general way so as to guarantee the occurrence of a Hopf bifurcation as $\lambda$ increases. More precisely, we wish three conditions to be satisfied simultaneously:

$$\lambda_H(\mathcal{T}^2, \sigma) < \lambda_c(\mathcal{T}^2);$$

$$\mathcal{T}^2(1 - \sigma)/(1 + \sigma) > \pi^4(A_i + 1)^3;$$

$$M_i = 1;$$

where the sequence $0 < A_1 < A_2 < ...$ is given

$$\{A_j| j = 1, 2, ...\} = \{a_i^2/\ell_1^2 + a_2^2/\ell_2^2| a_i \text{ integral, not both 0}\},$$

and the index $i$ is determined by

$$[((\sigma + 1)/\sigma)^2 \mathcal{T}^2_{i-1} < \mathcal{T}^2 < [((\sigma + 1)/\sigma)^2 \mathcal{T}^2_i,$$

and where, finally, $\mathcal{T}^2_i$ is given by (2.30).
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What we seek is complicated but fairly elementary, and best explained graphically. We consider two functions

\begin{align*}
\ell_H(\alpha^2) &= 2(\sigma + 1) \{ \pi^4(\alpha^2 + 1)^3 + [\sigma/(\sigma + 1)]^2 \mathcal{F}^2 \} / \alpha^2 \\
\ell_c(\alpha^2) &= [\pi^4(\alpha^2 + 1)^3 + \mathcal{F}^2] / \alpha^2.
\end{align*}

(3.6) (3.7)

![Graph of \( \ell = \ell_H(\alpha^2) \) and \( \ell = \ell_c(\alpha^2) \), illustrating the locations of \( \Lambda_c, \Lambda_H, \alpha_H^2, \alpha_c^2 \), \( \alpha_H^2 \) and \( \beta^2 \).]

The absolute minima of \( \ell = \ell_H(\alpha^2) \) and \( \ell = \ell_c(\alpha^2) \) occur at \( \alpha^2 = \sigma_H^2 \) and \( \alpha^2 = \alpha_c^2 \) respectively, where

\[ \pi^4(2\alpha_H^6 + 3\alpha_H^4 - 1) = [\sigma/(\sigma + 1)]^2 \mathcal{F}^2, \]

(3.8) and

\[ \pi^4(2\alpha_c^6 + 3\alpha_c^4 - 1) = \mathcal{F}^2. \]

(3.9)

Let \( \Lambda_H = \ell_H(\alpha_H^2) \) and \( \Lambda_c = \ell_c(\alpha_c^2) \) be these two minima. We may guarantee

\[ \Lambda_H < \Lambda_c, \]

(3.10) by imposing a suitable condition on \( \mathcal{F}^2 \) and \( \sigma \).

**Proposition 3.1.** Let \( \sigma^* \) be the positive root of

\[ 8\sigma^4 - \sigma - 1 = 0; \]

(3.11)

we have \( \sigma^* \approx 0.6766 \ldots \). Let

\[ \alpha(\sigma)^2 = \frac{3}{2} \frac{\sigma + 1 - 2\sigma^2}{\sigma + 1 - 2\sigma^2[2(\sigma + 1)]^\frac{1}{2}} - 1, \]

(3.12)

then

\[ 0 < \sigma < \sigma^*, \]

(3.13) and

\[ \mathcal{F}^2 > \pi^4[(\sigma + 1)/\sigma]^2[2\alpha(\sigma)^6 + 3\alpha(\sigma)^4 - 1], \]

(3.14)

together imply that \( \Lambda_H < \Lambda_c \).
Now, it is clear from (3.8) and (3.9) that always
\[ \alpha_H^2 < \alpha_c^2. \] (3.15)
Thus, the graphs of \( \ell = \ell_H(\alpha^2) \) and \( \ell = \ell_c(\alpha^2) \) may be pictured as in figure 1.
We define \( \beta^2 \) as the picture suggests, by requiring that
\[ \alpha_H^2 < \beta^2, \] (3.16)
and
\[ \ell_H(\beta^2) = \Lambda_c. \] (3.17)

**Proposition 3.2.** Suppose that \( \mathcal{T}^2 \) and \( \sigma \) satisfy (3.14) and suppose that in addition
\[ 0 < \ell_2 < \ell_1, \] (3.18)
\[ \alpha_H^2 \leq 1/\ell_1^2 < \beta^2, \] (3.19)
then (3.1) and (3.3) hold,
\[ \lambda_H(\mathcal{T}^2, \sigma) < \lambda_c(\mathcal{T}^2), \] (3.1)
and
\[ M_i = 1. \] (3.3)
The justification of this proposition is immediate. Equation (3.18) implies that
\[ A_1 = 1/\ell_1^2, \] (3.20)
and
\[ 1/\ell_1^2 < A_j \quad \text{for} \quad j = 2, 3, \ldots. \] (3.21)
The first half of (3.19), together with the fact that \( \ell_H(\alpha^2) \) is monotone increasing for \( \alpha_H^2 < \alpha^2 \) implies that
\[ \lambda_H(\mathcal{T}^2, \sigma) = \ell_H(1/\ell_1^2). \] (3.22)
The second half of (3.19), together with the monotonicity above, imply that
\[ \ell_H(1/\ell_1^2) < \Lambda_c. \] (3.23)
But the minimality of \( \Lambda_c \) implies
\[ \Lambda_c \leq \min \{\ell_c(A_j^2) | j = 1, 2, \ldots\} = \lambda_c(\mathcal{T}^2). \] (3.24)
As for \( M_i = 1 \), we have just seen that we may take \( i = 1 \), and since
\[ 1/\ell_1^2 > 1/\ell_1^2, \] (3.25)
there is exactly one way to write
\[ 1/\ell_1^2 = a_1^2/\ell_1^2 + a_2^2/\ell_2^2, \] (3.26)
with \( a_i \) as restricted in (2.25).
We still have to see that we may obtain (3.2) and (3.3) as well as the conditions above. Condition (3.2) is
\[ \mathcal{T}^2(1 - \sigma)/(1 + \sigma) > \pi^4(1/\ell_1^2 + 1)^3, \] (3.27)
and there will exist \( \ell_1^2 \) satisfying (3.19) and (3.27) provided that
\[ \mathcal{T}^2(1 - \sigma)/(1 + \sigma) > \pi^4(\alpha_H^2 + 1)^3. \] (3.28)
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**Proposition 3.3.** If $0 < \sigma^2 < \frac{2}{3}$, then the condition

$$\mathcal{F}^2 > \frac{27\pi^4(1-\sigma^2)(1+\sigma)^4}{(2-3\sigma^2)^3},$$

(3.29)

is equivalent to (3.28), which we write as

$$\alpha_H^2 < (1/\pi^4)\mathcal{F}^2[(1-\sigma)/(1+\sigma)]^{\frac{1}{2}} - 1.$$  

(3.30)

The justification of this proposition is a simple calculation, which we omit.

Finally, condition (3.5) is the condition that $A_i = A_1$ is the only $A_j$ for which $t_H(A_j) = \lambda(\mathcal{F}^2, \sigma)$. Using the conventions of §2, we have that (3.5) is equivalent to

$$0 < \mathcal{F}^2 < [(\sigma+1)/\sigma]_2^2\mathcal{F}^2.$$

(3.31)

Using (2.20), we see that $\alpha_H^2 \leq 1/\ell_1^2$ implies that

$$\mathcal{F}^2 > \pi^4(2\alpha_H^6 + 3\alpha_H^4 - 1) = [\sigma/(\sigma+1)]^2\mathcal{F}^2,$$

(3.32)

from which (3.31) follows.

We may collect these observations in a single theorem which reads as follows.

**Theorem 3.4.** Let $0 < \sigma < \sigma^*$, and then let

$$\mathcal{F}^2 > \pi^4\left(\frac{\sigma+1}{\sigma}\right)^2 \max\left\{2\alpha(\sigma)^6 + 3\alpha(\sigma)^4 - 1, \frac{27\pi^4\sigma^2(1-\sigma^2)(1+\sigma)^2}{(2-3\sigma^2)^3}\right\},$$

(3.33)

then $\beta^2$ is defined, and

$$\alpha_H^2 < \min\{\beta^2, (1/\pi^4)\mathcal{F}^2[(1-\sigma)/(1+\sigma)]^{\frac{1}{2}} - 1\} = \gamma^2.$$  

(3.34)

Let $0 < \ell_2 < \ell_1$ with

$$\alpha_H^2 \leq 1/\ell_1^2 < \gamma^2.$$  

(3.35)

Then (3.1)–(3.5) hold.

All the quantities in theorem 3.4 may be rapidly and explicitly computed with, say, a good hand calculator. For the reader’s convenience, we point out the simple formulae

$$\alpha_c = 3\pi^4[\frac{1}{2} + \cosh(\frac{1}{3}\arccosh(1 + 2\mathcal{F}^2/\pi^4))],$$

(3.36)

$$\beta^2 = -1 + \left[\frac{2}{3(\sigma+1)}\right]^{\frac{1}{2}} \arccosh\left[\frac{1}{2}\cosh\left(\frac{1}{224(\sigma+1)^3} \frac{2\sigma^2\mathcal{F}^2 + (\sigma+1)\Lambda_c}{\mathcal{F}^2 + \Lambda_c}\right)\right],$$

(3.37)

$$\alpha_H^2 = -\frac{1}{4} + \cosh\left\{\frac{1}{2}\arccosh(2(1 + (\mathcal{F}^2/\pi^4)\sigma)/[\sigma/(\sigma+1)]^2)\right\},$$

(3.38)

which result from solving the appropriate cubics, and which compute $\beta^2$ and $\alpha_H^2$ explicitly.

Finally, we remark that if we drop the parity condition (2.7), most of theorem 3.4 remains valid, excepting only the multiplicity result. Now $M_1 = 2$. Accordingly, the Joseph–Sattinger Hopf theorem will not apply, but we may hope that an invariant 2-torus will bifurcate, containing a period-locked closed orbit.
4. **Hopf Bifurcation**

In this section, we wish to show that we have sufficient hypotheses to apply the Hopf bifurcation theorem of Joseph & Sattinger. There are some slight differences between their context and ours, which we will have to take into account. To begin, we will use a 3-torus for our domain \( \Omega \) in place of their standard bounded domain in \( \mathbb{R}^3 \). Let \( \mathbb{Z} \) be the integers. Then allow \( \mathbb{Z}^3 \) to operate on \( \mathbb{R}^3 \) by means of translations,

\[
(x_1, x_2, x_3) \rightarrow (x_1 + 2n_1, x_2 + 2n_2, x_3 + 2n_3).
\]

The \( \Omega \) is the quotient of \( \mathbb{R}^3 \) by this action. Now let \( \tilde{L}_2^2(\Omega) \) be the Hilbert space of 4-fields \( u = (u_1, u_2, u_3, u_4) \) on \( \Omega \),

\[
\tilde{L}_2^2(\Omega) = \left\{ u : \int_{\Omega} (u_1^2 + u_2^2 + u_3^2 + u_4^2) \, dx = \|u\|^2 < \infty \right\}.
\]

The constant 4-vector fields are elements of this space, and we define their orthogonal complement to be

\[
\tilde{L}_2^2(\Omega) = \left\{ u \in \tilde{L}_2^2(\Omega) | u \perp c, \, c \text{ constant} \right\}.
\]

Let \( L_2^2(\Omega)_b \) be the subspace of \( L_2^2(\Omega) \) consisting of real vector fields which are weakly divergent in their first three components; that is, the functions \( u_i \) are real and

\[
\int_{\Omega} (u_1 \partial_1 p + u_2 \partial_2 p + u_3 \partial_3 p) \, dx = 0,
\]

for all \( p \) in \( C^1 \). We denote the orthogonal projection of \( L_2^2(\Omega) \) onto \( L_2^2(\Omega)_b \) by \( P \). We use \( C^{k+2\alpha} \) and \( C^{k+2\alpha, \ell+\alpha} (\Omega \times [0, T]) \) to denote the Banach spaces of 4-vector fields (instead of 3-vector fields) with Hölder norms \( \| \cdot \|_{k+2\alpha} \) on \( \Omega \) and \( \| \cdot \|_{k+2\alpha, \ell+\alpha} \) on \( \Omega \times [0, T] \) respectively. We need made no assumption on the smoothness of \( \partial \Omega \) since \( \partial \Omega = \phi \). The index \( k + 2\alpha \) refers to the number of times the vector field is to be differential and \( \ell + \alpha \) refers to the power of \( \| \cdot \| \) to be used, and \( 0 < \alpha < \frac{1}{2} \). Then the projection \( P \) is continuous in the Hölder norms.

Define an operator \( A \) in \( L_2^2(\Omega) \) by setting

\[
\mathcal{D}(A) = \{ u \in L_2^2(\Omega) | (\Delta u_1, \Delta u_2, \Delta u_3, \Delta u_4) \in \mathcal{L}_2^2(\Omega) \},
\]

and

\[
Au = P \begin{pmatrix} -\sigma \Delta u_1 \\ -\sigma \Delta u_2 \\ -\sigma \Delta u_3 \\ -\sigma \Delta u_4 \end{pmatrix}.
\]

Then clearly \( Au = 0 \) implies \( u = 0 \). Moreover, there exist constants \( K_1 \) and \( K_2 \) (for a given \( k \)) such that \( u \in C^{k+2\alpha}(\Omega) \) implies

\[
K_1 \|u\|_{k+2\alpha} \leq |Au|_{k+2\alpha} \leq K_2 \|u\|_{k+2\alpha}.
\]
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The Navier–Stokes equations (2.4) we rewrite in $H$ as

$$\frac{\partial u}{\partial t} = -L(\lambda) u - N(u, u),$$

where

$$L(\lambda) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = P \begin{pmatrix} -\sigma \Delta u_1 - \sigma I u_2 \\ -\sigma \Delta u_2 + \sigma I u_1 \\ -\sigma \Delta u_3 - u_4 \\ -\Delta u_4 - \sigma \lambda u_3 \end{pmatrix},$$

and

$$N(v, v) = P \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

Notice that if $v \in L^2(\Omega) \cap C^2(\Omega)$ then the vector function within the brackets of either (4.8) or (4.10) is again in $L^2(\Omega)$ so that the right side of (4.8) carries a dense domain into itself. We will say that a closed subspace of $L^2(\Omega)$ with that property is invariant. More precisely, suppose that $\mathcal{K}$ is a closed linear subspace of $L^2(\Omega)$ such that

(i) $\mathcal{K} \cap C^2(\Omega)$ is dense in $\mathcal{K}$;
(ii) if $v \in \mathcal{K}$ and the distributional derivatives $\Delta v$ and $\partial_i v$ are functions, and $L(\lambda) v \in L^2(\Omega)$ and $N(v, v) \in L^2(\Omega)$, then $L(\lambda) v \in \mathcal{K}$ and $N(v, v) \in \mathcal{K}$;

Then we will say that $\mathcal{K}$ is invariant. The argument of Joseph & Sattinger cannot distinguish between an invariant subspace and the whole space, consequently their argument will work in an invariant subspace. Notice that $L_2(\Omega)_0$ is invariant.

Finally, in $C^{k+2, \ell+\alpha}(\Omega \times [0, 2\pi]) \cap [0, 2\pi]$, let $P^{k+2, \ell+\alpha}_{2\pi}$ be the subspace of $2\pi$-periodic vector fields.

Now we are in the context of Joseph & Sattinger (1972), and their argument goes through to prove the following theorem.

**Hopf Theorem** (Joseph & Sattinger 1972). *Suppose that the characteristic values of $L(\lambda)$ have negative real part in $\mathcal{K}$ for $\lambda < \lambda_0$ and suppose further

(i) $L(\lambda_0)$ has exactly one pair of conjugate purely imaginary eigenvalues $\pm iP_1$;
(ii) each of these has multiplicity one;
(iii) if $p(\lambda)$ is the eigenvalue closest to $iP_1$ for $\lambda$ near $\lambda_0$, then $(\text{d Re } P(\lambda)/\text{d} \lambda)_{\lambda_0} \neq 0$;

then there exist convergent power series in $\epsilon$ as follows.

$$f(\epsilon) = \sum_{n=0}^{\infty} f_n \epsilon^n, \quad f_n \in \mathbb{R}, \quad f_0 = P_1,$$

$$\lambda(\epsilon) = \sum_{n=0}^{\infty} \lambda_n \epsilon^n, \quad \lambda_n \in \mathbb{R},$$

$$v(\epsilon) = \sum_{n=1}^{\infty} v_n \epsilon^n, \quad v_n \in P^{k+2, \ell+\alpha}_{2\pi},$$
such that

(i) for $\lambda = \lambda(\epsilon)$,

$$u(\epsilon)(x, t) = v(\epsilon)(x, f(\epsilon)t)$$

(4.14)

is a periodic solution of (4.8);

(ii) for $\lambda$ near $\lambda_0$ the only periodic solution of (4.8) near the origin is (4.14) for $\epsilon$ such that $\lambda = \lambda(\epsilon)$;

(iii) the coefficients $\lambda_{2n+1}$ and $f_{2n+1}$ are all zero;

(iv) if $\lambda_{2k}$, the first non-vanishing coefficient in (4.12), is positive, then the periodic solution (4.14) for $0 < \epsilon$ sufficiently small is asymptotically stable;

(v) if $\lambda_{2k}$ is negative then (4.14) is unstable.

It follows immediately that near the origin and $\lambda_0$ only three cases are possible: the periodic solutions occur only when $\lambda = \lambda_0$; they occur only for $\lambda < \lambda_0$; or, they occur only when $\lambda_0 < \lambda$. The latter two cases are called the subcritical and supercritical cases respectively, and (iv) and (v) imply that in the subcritical case the branching periodic solutions are unstable while in the supercritical case they are asymptotically stable.

In our particular case of the rotating Bénard problem we suppose that $R$ is a rigid motion of Euclidean space commuting with the translations, that define $\Omega$. Then $R$ defines a rigid motion $\hat{R} : \Omega \rightarrow \Omega$, and a unitary transformation $R^* : L_2(\Omega) \rightarrow L_2(\Omega)$ by continuous extension of

$$(R^*u)(x) = \begin{pmatrix} R_{11} & 0 & \ldots & 0 \\ 0 & R_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & R_{nn} \end{pmatrix} \begin{pmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{pmatrix},$$

(4.15)

where $u \in L_2(\Omega) \cap C^2(\Omega)$. It is easy to see that $R^* : L_2(\Omega) \rightarrow L_2(\Omega)$. Now, following the review of Kirchgassner (1975), we observe that we may define an invariant subspace by

$$I(R) = \{u \in L_2(\Omega) : R^*u = u \}.$$  

(4.16)

Using this construction we may obtain the subspace of $L_2(\Phi)$ satisfying free boundary conditions (2.5) at $z = 0$ and $z = 1$ by observing that it is simply the invariant subspace $I(R_0)$ where

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

the reader may convince himself by considering finite trigonometric series.

The function space $I(R_0)$ is not suitable for an application of the Hopf theorem of Joseph & Sattinger because, as may easily be seen every characteristic value of $L(\lambda)$ occurs with multiplicity greater than one. Accordingly we seek a smaller invariant function space. Let

$$R_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$
Then the intersection
\[ \mathcal{H} = I(R_0) \cap I(R_1), \]
is an invariant subspace. Notice that the parity condition (2.7) is equivalent to \( u \in I(R_1) \). The calculations of §3 now lead to a Hopf bifurcation. Specifically, as a corollary to theorem 3.4 we obtain the following theorem, using the notation of §3.

**Theorem 4.1.** Let \( 0 < \sigma < \sigma^* \), and then let
\[ \mathcal{F}^2 > \pi^4 \left( \frac{\sigma + 1}{\sigma} \right)^2 \max \left\{ 2\alpha(\sigma)^2 + 3\alpha(\sigma)^4 - 1, \frac{27\pi^4\sigma^2(1 - \sigma^2)(1 + \sigma)^2}{(2 - 3\sigma^2)^3} \right\}. \]
Let \( 0 < \ell_2 < \ell_1 \) with
\[ \alpha_h^2 \leq \frac{1}{\ell_2^2} < \min \{ \beta^2, (1/\pi^4)\mathcal{F}^2[(1 - \sigma)/(1 + \sigma)]^3 - 1 \}, \]
then a Hopf bifurcation occurs in the system defined by (2.1)–(2.7) as \( \lambda \) crosses \( \lambda_H(\mathcal{F}^2, \sigma) \).

Moreover, in the expansions (4.11)–(4.13) we have
\[ f_0^2 = \sigma^2[\mathcal{F}^2(1 - \sigma)/(1 + \sigma) - \pi^4(A_i + 1)^2](A_i + 1), \]
by (2.29); and
\[ \lambda_0 = \lambda_H(\mathcal{F}^2, \sigma). \]

In theorem 4.1, the coarse conditions for a Hopf bifurcation are verified for a pattern of roll type since \( a_g = 0 \). We may ask whether these conditions may be verified for the other standard types of patterns, squares, triangles or hexagons. The answer is that for these other types the multiplicity one condition is never satisfied. The reason is that for these other types the multiplicity one condition is never satisfied. The reason is that for these other types one must have both \( a_1 > 0 \) and \( a_2 \neq 0 \). But since \( a_1^2/\ell_1^2 + a_2^2/\ell_2^2 = a_1^2/\ell_1^2 + (-a_2)^2/\ell_2^2 \), the multiplicity of the purely imaginary characteristic values will be even by proposition 2.2. However, one can say that, aside from countably many exceptional ratios \( \ell_1/\ell_2 \), that multiplicity will be exactly two. In this case two independent Hopf bifurcations may be taking place with varying combinations of criticality and stability. However, more generally one may expect that a ‘double Hopf bifurcation’ takes place, producing an invariant 2-torus instead of a periodic solution. One possibility is for the invariant torus to be supercritical and stable. On such a torus there appear to be two periods. However, if the flow on the torus is structurally stable, as one might well expect in a physical situation, then the theorem of Peixoto (1962) implies that that flow is determined in the obvious way by a finite number of attracting periodic solutions and an equal number of repelling periodic solutions on that torus. The two apparent periods will have the same rational ratio on each of these periodic solutions; that is, ‘phase locking’ occurs. Physically one would observe then a transition from the conducting solution to one of the attracting phase locked periodic solutions. Thus, in the case of squares, triangles or hexagons a single Hopf bifurcation does not take place, although a bifurcation to a period solution may well occur. On the other hand, when the multiplicity of the imaginary characteristic value is 4, the theory of transition to turbulence of Ruelle & Takens (1971 a, b) suggests that the conducting solution
may well lose stability to a branching 4-torus, in turn contains a strange attractor. Physically one would observe a transition from the conducting solution to non-deterministic or perhaps turbulent behaviour.

Since it is not yet known how to detect strange attractors, nor even how to guarantee structural stability, we can only speculate as above about the observable effects of the multiplicity of the imaginary characteristic value. However, this speculation strongly suggests that although multiplicity 2 may still lead to stable periodic behaviour, multiplicity 4 may already lead to turbulent behaviour. The only certain information is that multiplicity 1 in the Hopf theorem implies periodic behaviour. Thus the multiplicity may be a purely technical quantity whose value one is necessary to the proof of the Hopf theorem, or the multiplicity may be a physically meaningful quantity whose value actually controls the transition to turbulence. We adopt the Ruelle–Takens point of view here to argue that until it is proved otherwise, the latter may well be the case, and that even in a practical hard-nosed computation one is not entitled to conclude that a transition to periodic behaviour takes place until he has checked that the multiplicity actually equals 1.

5. The algorithm for criticality

As Joseph & Sattinger show, if a Hopf bifurcation takes place at \( \lambda = \lambda_0 \), then the periodic solutions for \( \lambda \) near \( \lambda_0 \) will occur only for \( \lambda < \lambda_0 \) or only for \( \lambda_0 < \lambda \) or only for \( \lambda_0 = 0 \). In the case \( \lambda < \lambda_0 \) they show that the solutions are unstable; this is called the subcritical case. In the case \( \lambda_0 < \lambda \) they show that the solutions are asymptotically stable; this is called the supercritical case.

Physical common sense tells us that unstable periodic solutions are not physically detectable while asymptotically stable ones are detectable. Thus the determination of criticality is crucial to applications. To settle criticality, Joseph & Sattinger present an alternative proof of part of their Hopf theorem. This proof is basically an algorithm to compute the successive terms in a power series expansion of the periodic solution with respect to a certain parameter \( \epsilon \). The power series involved converge as a consequence of their Hopf theorem; alternatively, the power series may be proved convergent directly. As Joseph & Sattinger point out, this technique is the one used by Hopf in the finite dimensional case. However, they expand the frequency, rather than the period, of the periodic solution in powers of \( \epsilon \) and from the computational point of view this change makes their algorithm more tractable. The same expansion also appears as a ‘finite amplitude expansion’ in the work of Veronis (1959) in slightly different language; he too expands the frequency rather than the period.

The algorithm may be described most efficiently for a system

\[
\frac{\partial u}{\partial t} + L(\lambda) u + N(u) = 0, \tag{5.1}
\]

which undergoes a Hopf bifurcation at \( \lambda = \lambda_0 \) from the solution \( u = 0 \); here \( L(\lambda) = L_0 + \alpha L_1 \) with \( L \) linear operators (more generally \( L = \Sigma \alpha L_\alpha \)) and \( N(u) \) is a
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quadratic operator (more generally, a sum of operators homogeneous in \( u \) of degree \( \geq 2 \)). We seek solutions with \( u(t) \) in a real Hilbert space \( \mathcal{H} \) which is the closure of a pre-Hilbert subspace \( H \) with the property that \( L(\lambda) \) and \( N \) are defined on \( H \) and carry it into itself. Let \( \langle , \rangle \) and \( | | \) denote the inner product and norm of \( \mathcal{H} \). As usual, we extend the inner product \( \langle , \rangle \) and norm \( | | \) to a Hermitian inner product and norm on the complexification \( \mathbb{C} \otimes \mathcal{H} = \mathcal{H} \oplus i\mathcal{H} \) by setting \( \langle \xi + i\zeta, \mu + iv \rangle = \langle \xi, \mu \rangle + i\langle \zeta, \mu \rangle - i\langle \xi, v \rangle + \langle \zeta, v \rangle \); we use the same notation for the Hermitian inner product and norm. The periodic solution(s) \( u(t) \) of the Hopf bifurcation in particular will be periodic with frequency \( f \); since this frequency varies with \( \lambda \) we re-scale \( t \) so that the frequency becomes 1,

\[
\tilde{u}(t) = \tilde{u}(ft),
\]

where now \( \tilde{u}(\tau) \) has period \( 2\pi \) and frequency 1. Then (4.1) becomes

\[
f \partial \tilde{u}/\partial \tau + L(\lambda) \tilde{u} + N(\tilde{u}) = 0.
\]

With all the relevant periods now the same \( 2\pi \), we may introduce the Hilbert space

\[\mathcal{H}_{2\pi} = \left\{ u : \mathbb{R} \to \mathcal{H} | u \text{ measurable}, u(t + 2\pi) = u(t) \text{ almost everywhere}, \int_0^{2\pi} u(t)^2 \, dt < \infty \right\},\]

and the pre-Hilbert subspace

\[H_{2\pi} = \{ u | u = \text{real} \left( \sum_{n \text{ finite}} e^{in\tau} u_n, u_n \in H \} \}.
\]

Let \( \langle , \rangle \) and \( || | \) denote the inner product and norm of \( \mathcal{H}_{2\pi} \); we extend these to the complexification as above without change of notation. We seek a solution \( \tilde{u} \) of (4.3) as a convergent power series

\[
\tilde{u} = e \sum_{n=0}^{\infty} e^n \tilde{u}_n,
\]

\[
\lambda = \sum_{n=0}^{\infty} e^n \lambda_n,
\]

\[
f = \sum_{n=0}^{\infty} e^n f_n,
\]

with \( \tilde{u}_n \in H_{2\pi} \), \( \lambda_n \in \mathbb{R} \) and \( f_n \in \mathbb{R} \). In our case the work of Joseph & Sattinger (1972) will guarantee convergence. Here we are interested in the mechanics of carrying out the program above. Inserting (5.6)–(5.8) in (5.3) and comparing coefficients of \( e^n \), we obtain for \( n = 1 \)

\[
f_0 \partial \tilde{u}_0/\partial \tau + L(\lambda_0) \tilde{u}_0 = 0,
\]

and for \( n > 1 \),

\[
f_0 \partial \tilde{u}_n/\partial \tau + L(\lambda_0) \tilde{u}_n = -f_n \partial \tilde{u}_0/\partial \tau - \lambda_n L_1 \tilde{u}_0 + F_n,
\]

where

\[
F_n = - \sum_{r+s=n, 0<r<n} f_r \partial \tilde{u}_s/\partial \tau + \lambda_s L_1 \tilde{u}_s - \sum_{r+s=n-1} B(\tilde{u}_r, \tilde{u}_s),
\]
and $B(u, v)$ is a bilinear form such that $B(u, u) = N(u)$. Now we attempt to find $\tilde{u}_n$, $\lambda_n$ and $f_n$ recursively, starting with $\lambda_0$ the value of $\lambda$ at which the Hopf bifurcation occurs and $f_0 > 0$ such that $\pm if_0$ are the two purely imaginary eigenvalues of $L(\lambda_0)$. Let $\zeta$ and $\bar{\zeta}$ in $C \otimes \mathcal{H}$ be the corresponding eigenvectors. Then $e^{-i\tau}\zeta$ and $e^{i\tau}\bar{\zeta}$ in $C \otimes \mathcal{H}_{2\pi}$ are independent solutions of the linear equation (5.9). For $u_0$ we may take any non-zero vector in the two dimensional real subspace of the span of $e^{-i\tau}\zeta$ and $e^{i\tau}\bar{\zeta}$. Thus

$$\tilde{u}_0 = a e^{-i\tau}\zeta + \bar{a} e^{i\tau}\bar{\zeta},$$

(5.12)

where $a$ is any non-zero complex number.

To find a solution $\tilde{u}_n$ of (5.10) for $n \geq 1$ we need to consider the adjoint of $f_0 \partial/\partial\tau + L(\lambda_0)$,

$$(f_0 \partial/\partial\tau + L(\lambda_0))^* = -f_0 \partial/\partial\tau + L(\lambda_0)^*,$$

(5.13)

where $L(\lambda_0)^*$ denotes the adjoint of $L(\lambda_0)$ in $\mathcal{H}$ and therefore in $\mathcal{H}_{2\pi}$. Clearly the operator $f_0 \partial/\partial\tau + L(\lambda_0)$ carries its domain dom into the orthogonal complement $C$ of the kernel of $-f_0 \partial/\partial\tau + L(\lambda_0)^*$. With a suitable blanket assumption, we may conclude that $-f_0 \partial/\partial\tau + L(\lambda_0)^*$ also has a kernel of dimension 2 and that the map

$$f_0 \partial/\partial\tau + L(\lambda_0): \text{dom} \to C$$

(5.14)

is onto. Such a blanket assumption is contained in the following two conditions:

$$(i) \quad L(\lambda)** = L(\lambda) \text{ for all } \lambda;$$

$$(ii) \quad L(\lambda) \text{ has compact resolvent for all } \lambda.$$  

(5.15)

However, in our case in §5 below, we will use less sophisticated conditions, on the pre-Hilbert space $H_{2\pi}$ mentioned above. We will find a sequence of finite dimensional subspaces $H^m_{2\pi}$ of $H_{2\pi}$ such that

(i) $H^m_{2\pi} \subset H^{m+1}_{2\pi}$

(ii) $\bigcap_{m=0}^{\infty} H^m_{2\pi} = H_{2\pi}$;

(iii) $H^m_{2\pi}$ is invariant under

$$\partial/\partial\tau, \quad L(\lambda) \quad \text{and} \quad L(\lambda)^*$$

(5.16)

for all $\lambda$;

(iv) The kernels of $f_0 \partial/\partial\tau + L(\lambda_0)$ and $-f_0 \partial/\partial\tau + L(\lambda_0)^*$ are in $H^0_{2\pi}$;

(v) If $u, v \in H^{m-1}_{2\pi}$, then $B(u, v) \in H^m_{2\pi}$.

It follows immediately that if we have found $\tilde{u}_r \in H^r_{2\pi}$ for $r = 0, 1, \ldots, n - 1$ then $G_n \in H^m_{2\pi}$, where $G_n$ is the right side of (5.10). Moreover, now the restrictions to the finite dimensional vector space $H^m_{2\pi}$ of $f_0 \partial/\partial\tau + L(\lambda_0)$ and $-f_0 \partial/\partial\tau + L(\lambda_0)^*$ are mutually adjoint, and ordinary finite dimensional linear algebra tells us that the equation

$$f_0 \partial/\partial\tau + L(\lambda_0) \tilde{u}_n = G_n,$$

(5.17)

has a solution in $H^m_{2\pi}$ if and only if $G_n$ is normal to the kernel of $-f_0 \partial/\partial\tau + L(\lambda_0)^*$. Given $\tilde{u}_0, \ldots, \tilde{u}_{n-1}, f_0, \ldots, f_n$ and $\lambda_0, \ldots, \lambda_n$ we may find $\tilde{u}_n$ for $n \geq 1$ provided that
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\( (G_n, v) = 0 \). It follows from the simplicity of \( f_0 \) as an eigenvalue of \( L(\lambda_0) \) that the kernel of \( f_0 \partial/\partial t + L(\lambda_0) \) restricted to \( H_2^n \) is two dimensional. Accordingly, its adjoint, \( -f_0 \partial/\partial t + L(\lambda_0)^* \) restricted to \( H_2^n \), also has a two dimensional kernel on \( H_2^n \). But then an easy argument shows that \( -f_0 \partial/\partial t + L(\lambda_0)^* \) has a two dimensional kernel on \( H_2^n \) itself.

Now, subject either to (5.15) or (5.16), the necessary and sufficient condition for (5.10) to have a solution \( \tilde{u}_n \) is that \( \langle \tilde{u}_n, v \rangle = 0 \) for all \( v \in \text{kernel} (-f_0 \partial/\partial t + L(\lambda_0)^*) \), for any real basis \( v^*_\alpha \) and \( v^*_\beta \) of that kernel. We see that we may find \( \tilde{u}_n \) if and only if the two conditions

\[
\begin{align*}
f_n \langle \partial \tilde{u}_n/\partial t, v^*_\gamma \rangle &+ \lambda_n \langle L \tilde{u}_n, v^*_\gamma \rangle = \langle F_n, v^*_\gamma \rangle, \\
\gamma & = \alpha, \beta
\end{align*}
\]

(5.18)

hold.

Given \( \tilde{u}_0 \), ..., \( \tilde{u}_{n-1} \), \( f_0 \), ..., \( f_{n-1} \) and \( \lambda_0 \), ..., \( \lambda_{n-1} \), we may regard (5.18) as a pair of linear conditions on \( f_n \) and \( \lambda_n \). Joseph & Sattinger show that condition (iii) in the Hopf theorem above is exactly what is needed to conclude that these two linear conditions are independent, and so determine \( f_n \) and \( \lambda_n \) uniquely. Now we may solve for \( \tilde{u}_n \), uniquely if we impose the conditions

\[
\begin{align*}
\langle \tilde{u}_n, v^*_\alpha \rangle &= 0, \\
\langle \tilde{u}_n, v^*_\beta \rangle &= 0.
\end{align*}
\]

(5.19)

Comparing \( \ker (f_0 \partial/\partial t + L(\lambda_0)) \) with \( \ker (-f_0 \partial/\partial t + L(\lambda_0)^*) \) we see that at least in the case of (4.16) we may choose a basis \( \tilde{u}_0 = u_\alpha, u_\beta \) for the first kernel and then the basis \( v^*_\alpha, v^*_\beta \) for the second so that

\[
\begin{align*}
\langle u_\alpha, v^*_\alpha \rangle &= \langle u_\beta, v^*_\alpha \rangle = 1, \\
\langle u_\alpha, v^*_\beta \rangle &= \langle u_\beta, v^*_\beta \rangle = 0.
\end{align*}
\]

(5.20)

Then we see that for \( \tilde{u} \) as in (4.6) we have

\[
\varepsilon = \langle \tilde{u}, v^*_\alpha \rangle,
\]

(5.21)

to yield an interpretation of \( \varepsilon \) as a partial measure of the amplitude of \( \tilde{u} \) in the \( \tilde{u}_0 = u_\alpha \) direction. However, if one wishes merely to calculate the first non-vanishing \( \lambda_{2k} \) and to disregard the interpretation above, the normalization conditions (5.19) need not be imposed.

Finally we recall from (Joseph & Sattinger 1972) that the odd components \( \lambda_{2n+1} \) and \( f_{2n+1} \) all vanish.

### 6. Computation of criticality

We wish to apply the algorithm of §5 to the system (2.1)–(2.7). Each of the finite dimensional spaces \( H_2^n(\alpha) \) consisting of vector functions of the form (2.8) gives rise to a corresponding subspace \( K(\alpha, n) \) of \( L^2(\Omega) \) of §4, determined by the projection of the first 4-tuple of (2.8) into \( L^2(\Omega) \). The condition on \( U, V, W, T \) is simply that

\[
U \alpha_1 + V \alpha_2 + W_n = 0,
\]

(6.1)
so in fact $\mathcal{K}(\alpha, n) \subset \mathcal{H}(\alpha, n)$ and the components of members of $\mathcal{K}(\alpha, n)$ are trigonometric monomials. The space $H$ of §5 is the direct sum. Notice that $\mathcal{H}$ is the space $\mathcal{H} = I(R_0) \cap I(R_1)$ given by (4.17). Furthermore, the spaces $H^m_{2\pi}$ of §4 are given by

$$H^m_{2\pi} = \{ u | u = \text{real} \sum_{|k| \leq m} e^{ik\pi u_k}, u_k \in \oplus \{ K(\alpha, n), |\alpha_1| \leq m, |\alpha_2| \leq m, n \leq m \} \}$$

of the $\mathcal{K}(\alpha, n)$ and $\mathcal{H}$ is its completion in $L^2(\Omega)$. For ease of computation, we will not carry out the recursive algorithm of §5 in $\mathcal{H}$, but in the direct sum of the $\mathcal{H}(\alpha, n)$. To carry out the algorithm only the representatives in the $\mathcal{H}(\alpha, n)$ are needed of the elements in the $\mathcal{K}(\alpha, n)$, and the former are easier to write explicitly.

Now at last we may carry out the computation of criticality. Recall from the Hopf Theorem of Joseph–Saltzinger of §4 that for $\lambda_2 \neq 0$, criticality hinges on the sign of $\lambda_2$:

$$\begin{align*}
\lambda_2 > 0 & \text{ implies supercriticality and stability; } \\
\lambda_2 < 0 & \text{ implies subcriticality and instability.}
\end{align*} \tag{6.2}$$

For the system (2.1), equation (4.9) becomes

$$\begin{align*}
&f_0 \frac{\partial \tilde{u}_0}{\partial \tau} + \frac{1}{\sigma} \frac{\partial p_0}{\partial x} - \sigma \Delta \tilde{u}_0 + \sigma \mathcal{F} \tilde{v}_0 = 0, \\
&f_0 \frac{\partial \tilde{v}_0}{\partial \tau} + \frac{1}{\sigma} \frac{\partial p_0}{\partial y} - \sigma \Delta \tilde{v}_0 - \sigma \mathcal{F} \tilde{u}_0 = 0, \\
&f_0 \frac{\partial \tilde{w}_0}{\partial \tau} + \frac{1}{\sigma} \frac{\partial p_0}{\partial z} - \sigma \Delta \tilde{w}_0 - \theta_0 = 0, \\
&f_0 \frac{\partial \delta_0}{\partial \tau} - \Delta \delta_0 - \sigma \lambda_0 \tilde{w}_0 = 0, \\
&\frac{\partial \tilde{u}_0}{\partial x} + \frac{\partial \tilde{v}_0}{\partial y} + \frac{\partial \tilde{w}_0}{\partial z} = 0.
\end{align*} \tag{6.3}$$

The adjoint equation $-f_0 \frac{\partial v^*}{\partial \tau} + L(\lambda_0) v^* = 0$ is given by

$$\begin{align*}
&f_0 \frac{\partial u^*}{\partial \tau} + \frac{1}{\sigma} \frac{\partial p^*}{\partial x} + \sigma \Delta u^* + \sigma \mathcal{F} v^* = 0, \\
&f_0 \frac{\partial v^*}{\partial \tau} + \frac{1}{\sigma} \frac{\partial p^*}{\partial y} + \sigma \Delta v^* - \sigma \mathcal{F} u^* = 0, \\
&f_0 \frac{\partial w^*}{\partial \tau} + \frac{1}{\sigma} \frac{\partial p^*}{\partial z} + \sigma \Delta w^* + \sigma \lambda_0 \theta^* = 0, \\
&f_0 \frac{\partial \theta^*}{\partial \tau} + \Delta \theta^* + w = 0, \\
&\frac{\partial u^*}{\partial v} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} = 0.
\end{align*} \tag{6.4}$$

We will choose our external parameters $\sigma, \mathcal{F}, \ell_1 = 1$ and $\ell_2$ so that the solutions of (6.3) and (6.4) with $\lambda_0 = \lambda_n(\mathcal{F}^2, \sigma)$ lying in $\mathcal{K}(1, 0, 1)_{2\pi}$ are equal to the functions of
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$\tau$ of period $2\pi$ with values in $\mathbb{R}(1, 0, 1)$. Thus, these solutions will not depend on $y$, and we may add to our equations the following,

$$0 = \frac{\partial P}{\partial y} = \frac{\partial P^*}{\partial y} = \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial u^*}{\partial y} = \frac{\partial w}{\partial y} = \frac{\partial w^*}{\partial y}.$$  \hspace{1cm} (6.5)

In particular, we require that

$$\tilde{u}_0 = -\cos \tau \sin (\pi x) \cos (\pi z),$$  \hspace{1cm} (6.5)

then the last equation of (6.3) implies that

$$\bar{w}_0 = \cos \tau \cos (\pi x) \sin (\pi z).$$  \hspace{1cm} (6.7)

Using (2.3), we write

$$\tilde{v}_0 = V_0(\tau) \sin (\pi x) \cos (\pi z),$$  \hspace{1cm} (6.8)

and then the second equation of (6.3) becomes

$$f_0 dV_0(\tau) / d\tau + 2\pi^2 \sigma V_0(\tau) + \sigma \mathcal{F} \cos \tau = 0,$$  \hspace{1cm} (6.9)

so that we may write the solution

$$V_0(\tau) = -\frac{2\pi^2 \sigma^2 \mathcal{F} \cos \tau + \sigma \mathcal{F} f_0 \sin \tau}{4\pi^4 \sigma^2 + f_0^2};$$  \hspace{1cm} (6.10)

then (6.7) gives us $\tilde{v}_0$. In the same way we may obtain $\tilde{\theta}_0$. Thus we obtain

$$\begin{aligned}
\tilde{u}_0 &= -\cos \tau \sin (\pi x) \cos (\pi z), \\
\tilde{v}_0 &= V_{01} \cos \tau \sin (\pi x) \cos (\pi z) + V_{02} \sin \tau \sin (\pi x) \cos (\pi z), \\
\bar{w}_0 &= \cos \tau \cos (\pi x) \sin (\pi z), \\
\tilde{\theta}_0 &= \theta_{01} \cos \tau \cos (\pi x) \sin (\pi z) + \theta_{02} \sin \tau \cos (\pi x) \sin (\pi z), \\
\tilde{\varphi}_0 &= (P_{01} \cos \tau + P_{02} \sin \tau) \cos (\pi x) \cos (\pi z),
\end{aligned}$$  \hspace{1cm} (6.11)

with

$$\begin{aligned}
V_{01} &= -\frac{2\pi^2 \sigma^2 \mathcal{F}}{4\pi^4 \sigma^2 + f_0^2}; \\
V_{02} &= -\frac{\sigma \mathcal{F} f_0}{4\pi^4 \sigma^2 + f_0^2}, \\
\theta_{01} &= +\frac{2\pi^2 \sigma \lambda_0}{4\pi^4 + f_0^2}; \\
\theta_{02} &= -\frac{\sigma \lambda_0 f_0}{4\pi^4 + f_0^2}.
\end{aligned}$$  \hspace{1cm} (6.12)

Then the vector function given by (6.11) is in $\mathbb{R}(1, 0, 1)_{2\pi}$ and its projection on $\mathbb{R}(1, 0, 1)_{2\pi}$ is a solution of (5.9); alternatively, there is a pressure function $P_{00}$ which extends it to a solution of (6.3).

Now we turn to the adjoint equation (6.4); that is, we turn to the basis $v^{\alpha}_0, v^{\beta}_0$ of the kernel of the operator $-f_0 \partial / \partial \tau + L(\lambda_0)^*$ of $\S 5$. Here we need two real solutions which we distinguish by the subscripts $\alpha$ and $\beta$ as in $\S 5$. In the same way as above we find these solutions to be given by:

$$\begin{aligned}
\tilde{u}^*_0 &= -\cos \tau \sin (\pi x) \cos (\pi z); \\
\tilde{v}_0^* &= (V_{01}^* \cos \tau + V_{02}^* \sin \tau) \sin (\pi x) \cos (\pi z); \\
\bar{w}_0^* &= \cos \tau \cos (\pi x) \sin (\pi z); \\
\tilde{\theta}_0^* &= (\theta_{01}^* \cos \tau + \theta_{02}^* \sin \tau) \cos (\pi x) \sin (\pi z); \\
\tilde{\varphi}_0^* &= (P_{01}^* \cos \tau + P_{02}^* \sin \tau) \cos (\pi x) \cos (\pi z);
\end{aligned}$$  \hspace{1cm} (6.13)
and
\[
\begin{align*}
&u_\beta^* = -\sin \tau \sin (\pi x) \cos (\pi z), \\
v_\beta^* (V_\beta^* \cos \tau + B_\beta^* \sin \tau) \sin (\pi x) \cos (\pi z) \\
w_\beta^* = \sin \tau \cos (\pi x) \sin (\pi z), \\
\theta_\beta^* = (\theta_\beta^* \cos \tau + \theta_\beta^* \sin \tau) \cos (\pi x) \sin (\pi z), \\
\mathcal{P}_\beta^* = (P_\beta \cos \tau + P_\beta \sin \tau) \cos (\pi x) \cos (\pi z),
\end{align*}
\]
where
\[
\begin{align*}
V_{a1}^* &= \frac{2\pi^2 \sigma_0^2 \mathcal{F}}{4\pi^4 \sigma_0^2 + f_0^2}, \\
V_{a2}^* &= -\frac{\sigma_0 \mathcal{F} f_0}{4\pi^4 \sigma_0^2 + f_0^2}, \\
\theta_{a1}^* &= \frac{2\pi^2}{4\pi^4 + f_0^2}, \\
\theta_{a2}^* &= -\frac{\sigma_0 \mathcal{F} f_0}{4\pi^4 + f_0^2},
\end{align*}
\]
and
\[
\begin{align*}
V_{\beta1}^* &= \frac{\sigma_0 \mathcal{F} f_0}{4\pi^4 \sigma_0^2 + f_0^2}, \\
V_{\beta2}^* &= \frac{2\pi^2 \sigma_0^2 \mathcal{F}}{4\pi^4 \sigma_0^2 + f_0^2}, \\
\theta_{\beta1}^* &= \frac{f_0}{4\pi^4 + f_0^2}, \\
\theta_{\beta2}^* &= \frac{2\pi^2}{4\pi^4 + f_0^2}. \\
\end{align*}
\]

Notice that the 4-vectors defined by the first 4 components of (6.14) and (6.15) are already in \( L_q(\Omega_0) \). Now we turn to the computation of \( \lambda_1, f_1, \bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{\theta}_1 \) and \( \bar{P}_1 \). We begin by finding \( F_1 \) as given by (5.11) for the system (2.4)–(2.7). In our case (modulo gradients) we have
\[
B \begin{pmatrix} u & u' \\ v & v' \\ w & w' \\ \theta & \theta' \end{pmatrix} = \begin{pmatrix} \mathbf{\nu} \cdot \nabla u' \\ \mathbf{\nu} \cdot \nabla v' \\ \mathbf{\nu} \cdot \nabla w' \\ \mathbf{\nu} \cdot \nabla \theta' \end{pmatrix},
\]
where
\[
\mathbf{\nu} = \begin{pmatrix} u \\ v \\ w \end{pmatrix},
\]
thus
\[
F_1 = \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \end{pmatrix} = -\begin{pmatrix} \frac{\partial u_0}{\partial x} + w_0 \frac{\partial u_0}{\partial z} \\
\frac{\partial v_0}{\partial x} + w_0 \frac{\partial v_0}{\partial z} \\
\frac{\partial w_0}{\partial x} + w_0 \frac{\partial w_0}{\partial z} \\
\frac{\partial \theta_0}{\partial x} + w_0 \frac{\partial \theta_0}{\partial z} \end{pmatrix}.
\]
We find that

\[
F_1 = \begin{pmatrix}
-\pi \cos^2 \tau \sin (\pi x) \cos (\pi x) \\
(\pi V_{01} \cos^2 \tau + \pi V_{02} \cos \tau \sin \tau) \sin (\pi x) \cos (\pi x) \\
-\pi \cos^2 \tau \sin (\pi z) \\
-(\pi \theta_{01} \cos^2 \tau + \pi \theta_{02} \sin \tau \cos \tau) \sin (\pi z) \cos (\pi z)
\end{pmatrix}
\]  

(6.20)

Accordingly, modulo gradients, the right side \(G_1\) of (5.10) is given by

\[
G_1 = \begin{pmatrix}
\sin \tau \sin (\pi x) \cos (\pi z) - \pi \cos^2 \tau \sin (\pi x) \cos (\pi x) \\
(f_1 \sin \tau - f_1 V_{02} \cos \tau) \sin (\pi x) \cos (\pi z) \\
+ (\pi V_{01} \cos^2 \tau + \pi V_{02} \cos \tau \sin \tau) \sin (\pi x) \cos (\pi x) \\
-f_1 \sin \tau \cos (\pi x) \sin (\pi z) - \pi \cos^2 \tau \sin (\pi x) \cos (\pi z) \\
-(f_1 \theta_{01} \sin \tau - f_1 \theta_{02} \cos \tau + \lambda_1 \sigma \cos \tau) \sin (\pi x) \sin (\pi z) \\
+ (\pi \theta_{01} \cos^2 \tau + \pi \theta_{02} \sin \tau \cos \tau) \sin (\pi x) \cos (\pi z)
\end{pmatrix}.
\]  

(6.21)

Since the two 4-vector fields given by (6.13) and (6.14) are already orthogonal to gradients, conditions (5.18) become

\[
f_1[V_{x1}^* V_{y2} - V_{x2}^* V_{y1} + \theta_{x1}^* \theta_{y2} - \theta_{x2}^* \theta_{y1}] - \lambda_1 \sigma \theta_{x1}^* = 0,
\]

\[
f_1[-2 + V_{x1}^* V_{y2} - V_{x2}^* V_{y1} + \theta_{x1}^* \theta_{y2} - \theta_{x2}^* \theta_{y1}] - \lambda_1 \sigma \theta_{y1}^* = 0.
\]  

(6.22)

According to Joseph & Sattinger (1972), condition (iii) of the Hopf theorem guarantees that the determinant of the system (6.22) is non-zero. The unique solution of this system is given by

\[
f_1 = \lambda_1 = 0.
\]  

(6.23)

Thus we have to solve

\[
f_0 \frac{\partial \bar{u}_1}{\partial \tau} + \frac{1}{\sigma} \frac{\partial P_1}{\partial x} - \sigma \Delta \bar{u}_1 + \sigma \mathcal{F} \bar{u}_1 = -\frac{1}{4} \pi (1 + \cos 2\tau) \sin (2\pi x),
\]

\[
f_0 \frac{\partial \bar{v}_1}{\partial \tau} - \sigma \Delta \bar{v}_1 - \sigma \mathcal{F} \bar{u}_1 = \frac{1}{4} \pi [V_{01}(1 + \cos 2\tau) + V_{02} \sin 2\tau] \sin (2\pi x),
\]

\[
f_0 \frac{\partial \bar{w}_1}{\partial \tau} + \frac{1}{\sigma} \frac{\partial P_1}{\partial z} - \sigma \Delta \bar{w}_1 - \bar{\theta}_1 = -\frac{1}{4} \pi (1 + \cos 2\tau) \sin (2\pi z),
\]

\[
f_0 \frac{\partial \bar{\theta}_1}{\partial \tau} - \Delta \bar{\theta}_1 - \sigma \lambda_0 \bar{\omega}_1 = -\frac{1}{4} \pi [\theta_{01}(1 + \cos 2\tau) + \theta_{02} \sin 2\tau] \sin (2\pi z),
\]

\[
\frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{w}_1}{\partial z} = 0.
\]

All we require is one particular solution of (6.23) in our function space. To find it, the method of undetermined coefficients suffices, and we see that one such solution is given by

\[
\bar{u}_1 = 0,
\]

\[
\bar{v}_1 = (A + B \cos 2\tau + C \sin 2\tau) \sin (2\pi x),
\]

\[
\bar{w}_1 = 0,
\]

\[
\bar{\theta}_1 = (P + Q \cos 2\tau + R \sin 2\tau) \sin (2\pi z),
\]  

(6.24)
where

\[
\begin{align*}
A &= -\frac{1}{8} \pi \sigma \mathcal{F} \frac{1}{4 \pi^4 \sigma^2 + f_0^2}, \\
B &= \frac{1}{8} \pi \sigma \mathcal{F} \frac{(f_0^2 - 4 \pi^4 \sigma^2)}{(4 \pi^4 \sigma^2 + f_0^2)^2}, \\
C &= -\frac{1}{8} \pi \sigma^2 \mathcal{F} \frac{f_0}{(4 \pi^4 \sigma^2 + f_0^2)^2}, \\
P &= -\frac{1}{8} \pi \sigma \lambda_0 \frac{1}{4 \pi^4 + f_0^2}, \\
Q &= \frac{1}{8} \pi \sigma \lambda_0 \frac{f_0^2 - 4 \pi^4}{(4 \pi^4 + f_0^2)^2}, \\
R &= -\frac{1}{8} \pi \sigma^2 \lambda_0 \frac{f_0}{(4 \pi^4 + f_0^2)^2}.
\end{align*}
\]

(6.25)

Now, we turn to \( \lambda_2 \) and \( f_2 \), though we will not find \( f_2 \) explicitly. These two numbers may be found from foregoing information by applying condition (5.18). Recall the projection \( P : L_2(\Omega) \to L_2(\Omega) \), § 4. Let \( \mathbf{u}_i = (\bar{u}_i, \bar{v}_i, \bar{w}_i, \bar{\theta}_i) \) for \( i = 1, 2 \); and let \( \mathbf{v}_\gamma^* = (w^*_\gamma, v^*_\gamma, u^*_\gamma, \theta^*_\gamma) \) for \( \gamma = \alpha, \beta \). These are elements of \( L_2(\Omega) \). In particular \( P \mathbf{v}_\gamma^* = \mathbf{V}_\gamma^* \) and \( P \mathbf{u}_i = \mathbf{u}_i \). Now, condition (5.18) may be written

\[
f_2 \langle P \partial \mathbf{u}_0 / \partial \tau, \mathbf{v}_\gamma^* \rangle + \lambda_2 \langle P \mathcal{L} \mathbf{u}_0, \mathbf{v}_\gamma^* \rangle = \langle P \mathcal{F}, \mathbf{v}_\gamma^* \rangle, \quad \gamma = \alpha, \beta,
\]

(6.26)

where

\[
\mathcal{L}(u, v, w, \theta) = (0, 0, 0, -\sigma w),
\]

(6.27)

and

\[
\mathcal{F} = \left( 0, -\bar{u}_0 \frac{\partial \bar{v}_1}{\partial x} - \bar{v}_0 \frac{\partial \bar{u}_1}{\partial x}, 0, -\bar{u}_0 \frac{\partial \bar{\theta}_1}{\partial x} - \bar{v}_0 \frac{\partial \bar{\theta}_1}{\partial z} \right),
\]

(6.28)

and

\[
P \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau} P.
\]

(6.29)

Each inner product in (6.26) is of the form \( \langle P \xi, \mathbf{v}_\gamma^* \rangle \). But we have

\[
\langle P \xi, \mathbf{v}_\gamma^* \rangle = \langle \xi, P^* \mathbf{v}_\gamma^* \rangle = \langle \xi, P \mathbf{v}_\gamma^* \rangle = \langle \xi, \mathbf{v}_\gamma^* \rangle,
\]

(6.30)

so that we obtain the same condition on \( f_2 \) and \( \lambda_2 \) by dropping the \( P \),

\[
f_2 \langle \partial \mathbf{u}_0 / \partial \tau, \mathbf{v}_\gamma^* \rangle + \lambda_2 \langle \mathcal{L} \mathbf{u}_0, \mathbf{v}_\gamma^* \rangle = \langle \mathcal{F}, \mathbf{v}_\gamma^* \rangle, \quad \gamma = \alpha, \beta.
\]

(6.31)

We calculate the inner products and solve these two linear equations to conclude that

\[
\begin{align*}
\langle \partial \mathbf{u}_0 / \partial \tau, \mathbf{v}_\alpha^* \rangle &= \pi [V_{02} V_{a1} - V_{01} V_{a2} + \theta_{02} \theta_{a1} - \theta_{01} \theta_{a2}], \\
\langle \partial \mathbf{u}_0 / \partial \tau, \mathbf{v}_\beta^* \rangle &= -\pi [2 + V_{01} V_{a2} - V_{02} V_{a1} + \theta_{01} \theta_{a2} - \theta_{02} \theta_{a1}], \\
\langle \mathcal{L} \mathbf{u}_0, \mathbf{v}_\alpha^* \rangle &= -\pi \sigma \theta_{a1}, \\
\langle \mathcal{L} \mathbf{u}_0, \mathbf{v}_\beta^* \rangle &= -\pi \sigma \theta_{a2}^*, \\
\langle \mathcal{F}, \mathbf{v}_\alpha^* \rangle &= -\pi^2 [(A + \frac{1}{2} B) V_{a1} + \frac{1}{2} C V_{a2} - (P + \frac{1}{2} Q) \theta_{a1} - \frac{1}{2} R \theta_{a2}], \\
\langle \mathcal{F}, \mathbf{v}_\beta^* \rangle &= -\pi^2 [(A + \frac{1}{2} B) V_{a1} + \frac{1}{2} C V_{a2} - (P + \frac{1}{2} Q) \theta_{a1} - \frac{1}{2} R \theta_{a2}],
\end{align*}
\]

(6.32)
and then, of course, that

\[
\lambda_2 = \frac{\left\langle \frac{\partial u_0}{\partial t}, v_\alpha^* \right\rangle \left\langle \mathcal{F}, v_\beta^* \right\rangle - \left\langle \frac{\partial u_0}{\partial t}, v_\beta^* \right\rangle \left\langle \mathcal{F}, v_\alpha^* \right\rangle}{\left\langle \frac{\partial u_0}{\partial t}, v_\alpha^* \right\rangle \left\langle \mathcal{L} u_0, v_\beta^* \right\rangle - \left\langle \frac{\partial u_0}{\partial t}, v_\beta^* \right\rangle \left\langle \mathcal{L} u_0, v_\alpha^* \right\rangle}. \tag{6.33}
\]

7. Some specific cases of \( \lambda_2 \)

In principle, equation (6.33) expresses \( \lambda_2 \) explicitly in terms of \( \sigma \) and \( \mathcal{F} \). However, the equation is so complex that it may be enlightening to display the result in a greatly simplified case. First we take \( \alpha_H^2 = A_i = \text{minimizing } A_j \text{ for } \ell_H \). With the notation of § 3, recall that then for a Hopf bifurcation to occur, we must have

\[
\lambda_H^2(\mathcal{F}^2, \sigma) \lambda_0 = 6\pi^4(\sigma + 1) (\alpha_H^2 + 1)^2, \tag{7.1}
\]

\[
f^2 = \pi^4(\alpha_H^2 + 1) (2\pi^2_H - 1 - 3\sigma^2 \alpha_H^2), \tag{7.2}
\]

\[
\mathcal{F}^2 = \pi^4[(\sigma + 1)/\sigma]^2 (2\alpha_H^2 + 3\alpha_H^2 - 1). \tag{7.3}
\]

To apply these conditions to § 6, where \( \ell_1 = 1 \), we must have \( \alpha_H^2 = A_i = 1/\ell_1^2 = 1 \). Then (7.1)–(7.3) become

\[
\lambda_H = 24\pi^4(\sigma + 1), \tag{7.4}
\]

\[
f_0^2 = 2\pi^4(1 - 3\sigma^2), \tag{7.5}
\]

\[
\mathcal{F}^2 = 4\pi^4[(\sigma + 1)/\sigma]^2. \tag{7.6}
\]

We may replace condition (3.33) in theorem 3.4 with the weaker conditions (3.14) and (3.27) which have the same effect, to imply (3.1)–(3.5). Since \( \alpha_H^2 = 1 \), the condition

\[
\alpha(\sigma) < 1, \tag{7.7}
\]

implies (3.14). This condition is

\[
-128\sigma^5 - 92\sigma^4 + 12\sigma^3 + 13\sigma^2 + 2\sigma + 1 > 0, \tag{7.8}
\]

which holds for small \( \sigma \). In our case, condition (3.27) becomes

\[
1 > 3\sigma^2, \tag{7.9}
\]

which holds for \( \sigma < \frac{1}{3}/3 \). Thus for small \( \sigma \), the system (2.4)–(2.7) in which \( \alpha_H^2 = A_i = \ell_1^{-2} = 1 \) and \( 0 < \ell_2 < \ell_1 \), will undergo a Hopf bifurcation as \( \lambda \) passes \( 24\pi^4(\sigma + 1) \), provided \( \sigma \) is small enough.

Is this bifurcation critical or subcritical? To answer this question we must evaluate \( \lambda_2 \) in (6.33). To do so, we insert (7.4)–(7.6) in (6.12), (6.15), (6.16) and (6.25). Then we insert the result in (6.32), and finally, that result in (6.33). Since the determinant of (6.22) is never zero by Joseph & Sattinger 1972, and since that determinant is the denominator of (6.33), evaluation at a single point suffice to determine its sign, which is negative. The numerator may be shown to be always
positive in the entire range $0 < \sigma < \sigma^*$; thus $\lambda_2 < 0$ and the Hopf bifurcation determined above is always subcritical.

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