1 Independent Random Variables

Definition: two random variables \( X \) and \( Y \) are independent if the events \( X \leq t \) and \( Y \leq s \) are independent for all real numbers \( s \) and \( t \).

Equivalently:
- \( X \) and \( Y \) are independent if \( \Pr[X \leq t \text{ and } Y \leq s] = F_X(t) \cdot F_Y(s) \) for all \( s, t \).

- Prove that the indicator rv’s for two independent events \( A \) and \( B \) are independent.
- Prove that the number of heads in 4 flips of a fair coin is not independent of the indicator rv that the 4th flip is a head.

2 Expectation

The expectation of a random variable is a kind of weighted average, weighted according to the probability mass function. We define the expectation of a r.v. \( Y \):

\[
\mathbb{E}Y = \sum_{\text{outcomes } x \in S} p(x)Y(x)
\]

which is the same as

\[
\mathbb{E}Y = \sum_{t: \Pr[Y = t] \neq 0} P_Y(t)t
\]

- show that the two definitions are the same
- note: the expectation is not the most likely value \( Y \) takes, it is an average value.

Examples:
- We flip a fair coin. Let \( X = 10 \) if we get heads, \(-4\) if we get tails. What is \( \mathbb{E}X \)?
- Let \( Y \) be the number on a single roll of a die. What is \( \mathbb{E}Y \)?
- Let \( X \) be the sum of the numbers on two dice. What is \( \mathbb{E}Y \)?
- Let \( Z = 1 \) if we roll a 6 on a fair die, 0 otherwise. \( \mathbb{E}Z =? \)
- A hot dog man makes $100 if it is sunny, $50 if it is not. \( \Pr[sunny] = .8 \). What is his expected earnings?

3 Expectation and Fair Games

Expectation describes our intuitive idea of a fair game. Let’s say we play a game: you roll two dice and if two two numbers are the same you win $20 from me, and if they are different you lose \( $x \) dollars.

- What should \( x \) be for the game to be fair?
- Can you come up with a rigorous definition of a fair betting game? Are the games in a casino fair?
4 Expectation is Linear

A linear function is a function $f$ so that $f(ax + y) = af(x) + f(y)$.

Expectation is also linear:
$$E[aX + Y] = aE[X] + E[Y]$$

Prove it!

- What is the expectation of $I_A$, the indicator random variable for the event $A$?
- Let $Y$ be the number of heads in $n$ flips of a $p$-biased coin. We showed before that we can break up $Y = X_1 + \cdots + X_n$ where $X_i$ is the indicator r.v. that flip $i$ is a head. Use linearity to calculate $EY$.
- If $X$ and $Y$ are dependent, do we still have $E[X + Y] = E[X] + E[Y]$? If not, give a counterexample.
- If 53% of people prefer coke to pepsi, and we randomly sample 120 people, what is the expected number of people in our sample who prefer coke?
- If you repeat a fair game 100 times, is that also a fair game?

5 Variance

The variance of a random variable is defined as:
$$var(X) = E[(X - EX)^2]$$

this is the same as:
$$var(X) = E(X^2) - (EX)^2$$

prove that.

Calculate some variances:

- Let $Y = 7$ with probability 1. $var(Y) =$?
- Let $X_i$ be the indicator r.v. of a getting heads with a fair coin. $var(X_i) =$?
- $X_i$ is the indicator of a $p$-biased coin. $var(X_i) =$?
- If $X$ and $Y$ are independent, prove that $E[XY] = EXEY$. (*note: this does not go the other way*)

Final question to test everything you learned today:
Let $Y$ be the number of heads in 10 flips of a fair coin.

- What is $EY$?
- What is $var(Y)$?