1 Continuous Random Variables

1. Let $X$ be a random variable with density function $c_1 e^{-c_2 x}$ for $x \geq 0$. Assume $E[X] = 2$.
   
   • what are $c_1$ and $c_2$?
   • Write the CDF of $X$.
   • Calculate $Pr[3 \leq X \leq 5]$  

2. Let $X$ and $Y$ be independent normal random variables with means 1 and 3 and variances 4 and 9 respectively.
   
   • Find the mean and variance of $Q = 2X + Y$
   • Prove that $Q = 2X + Y$ has a normal distribution.

2 Covariance

1. Let $X_1, X_2$ be independent, each taking the values $-1, 1$ with probability 1/2. Let $Y_1 = 2X_1 - X_2$ and $Y_2 = 2X_2 - X_1$.
   
   • What is $cov(X_1, X_2)$?
   • What is $cov(Y_1, Y_2)$?
   • What is $cov(X_1, Y_1)$?

2. Express the variance of $X_1 + X_2 + X_3$ in terms of only variances and covariances, not expectations.

3. What is the covariance of the number of Red cards and the number of Black cards in a 5 card hand of poker?

   *answer:* Let $R$ and $B$ be the respective random variables. $R = R_1 + \cdots + R_5$ where $R_i$ is the indicator that the $i$th card is red. $ER = EB = 5/2$. We need to calculate $E[RB]$.

   $$E[RB] = E\left[\sum_{i,j} iR_iB_j]\right]$$

   $$= 5E[R_1B_1] + 20E[R_1B_2]$$

   by symmetry. $E[R_1B_1 = 0]$ and $E[R_1B_2] = \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{51}$. So $E[RB] = \frac{260}{51}$ and

   $$cov(R, B) = \frac{260}{51} - \frac{25}{4} = -\frac{235}{204}$$
3 Poisson Distributions

1. Let $X$ be Poisson with mean $\lambda$.
   - Find the value of $j$ that maximizes $\Pr[X = j]$.
   - For what value of $\lambda$ does $\Pr[X = 0] = \Pr[X = 1]$?

   **answer:**

   (a) $\Pr[X = k]$ increases as $k$ goes up from 0 until it hits its maximum, then it declines. (the function is unimodal). So we need to find the first $k$ so that $\Pr[X = k] > \Pr[X = k + 1].$

   
   $$\Pr[X = k] - \Pr[X = k + 1] = \frac{e^{-\lambda} \lambda^k}{k!} - \frac{e^{-\lambda} \lambda^{k+1}}{(k+1)!}$$

   $$= \frac{e^{-\lambda} \lambda^k}{k!} \left[ 1 - \frac{\lambda}{k + 1} \right]$$

   So if $k + 1 < \lambda$, $\Pr[X = k + 1] > \Pr[X = k]$. So the maximum occurs at $\lceil \lambda \rceil$.

   (b) $\lambda = 1$

4 Central Limit Theorem

1. Estimate as best you can (calculate a decimal) the probability that in 3600 flips of a fair coin you get less than 1860 heads.

5 Maximum Likelihood

1. Find the maximum likelihood estimator for the mean of a sequence of exponential random variables.

6 Parameter Estimation

Answers can be written in terms of probabilities of $Z$ or $T_n$.

1. You’re measuring the length of randomly captured butterflies, and you’ve caught 4, with lengths 4.8, 6.2, 5.4, 5.6 cm. What can you say about $\mu$, the true mean length of a random butterfly?

2. You take two separate random polls of voters in Idaho. In the first, 43 out of 100 people say they have a good impression of the Democratic candidate. In the second, 48 out of 100 say they have a good impression of the Republican candidate. You’re asked: “Do voters have a better impression of the Democrat or the Republican?” What can you say, and with what confidence?