Yes / no answers are not enough. Give some mathematical justification (which can be short and should not be an essay).

Answers like \( \binom{10}{3}(.5)^6 \) are fine - you don’t need to give a decimal.

The questions on the actual midterm will vary in difficulty and in general the more difficult questions will count less.

1

I tell you that a coin has probability of heads .75. You flip it 4 times and get no heads. Do you believe my claim? Justify your answer with statistics.

answer:

We use the statistical method. Say we set \( \alpha = .01 \). Our null hypothesis, \( H_0 \) is that the coin has probability of heads .75. The alternate hypothesis could either be that \( p \neq .75 \) or that \( p < .75 \). In either case, we calculate the probability under the null hypothesis that what we saw occurs or something more extreme. In this case 4 tails in 4 flips is the most extreme outcome (4 heads is not as extreme since under \( H_0 \) the coin is biased in favor of heads.) \( \Pr[4 \text{ tails}] = .25^4 = \frac{1}{256} \), so our p-value is less than .01 so we reject the null hypothesis and say that we don’t believe the coin has \( p = .75 \).

2

We model tomorrow’s weather as follows, with four possible outcomes: \((\text{sunny, warm}), (\text{sunny, cold})\), \((\text{rainy, warm}), (\text{rainy, cold})\) with probabilities \(.2, .6, .1, .1\) respectively. The salesman makes $30 selling umbrellas if it rains, but $0 on umbrellas if it is sunny. He also makes $60 selling gloves if it is cold, and $10 selling gloves if it is warm.

Let \( Y \) be the total amount of money he makes, \( U \) be the amount he makes on umbrellas, and \( G \) the amount he makes on gloves.

- Calculate \( \mathbb{E}Y, \mathbb{E}U, \mathbb{E}G \)
- Calculate \( \text{var}(Y) \)
- Are \( U \) and \( G \) independent?

answer:

\[
\mathbb{E}U = .2 \cdot 30 = 6 \\
\mathbb{E}G = .7 \cdot 60 + .3 \cdot 10 = 45 \\
\mathbb{E}Y = \mathbb{E}U + \mathbb{E}G = 51
\]

3

Let \( X \) be a random variable that always takes a value \( \geq 0 \). Prove that \( \mathbb{E}X \geq \Pr[X \geq 1] \).
Prove that if $X$ and $Y$ are independent random variables then

$$\mathbb{E}[XY] = \mathbb{E}X \cdot \mathbb{E}Y$$

answer:

$$\mathbb{E}[XY] = \sum_{t,s} ts \Pr[X = t, Y = s]$$

$$= \sum_{t,s} ts \Pr[X = t] \cdot \Pr[Y = s]$$

$$= \sum_t t \Pr[X = t] \mathbb{E}Y$$

$$= \mathbb{E}Y \cdot \mathbb{E}X$$

Let $X$ be an non-negative, integer-valued random variable. Prove that $\Pr[X \geq 1] \leq \mathbb{E}X$.

I pick a number from 1 to 1,000,000 at random, with equal probability.

- What’s the probability the last digit is a 3?
- What’s the probability the second-to-last digit is a 3?
- Are these events independent?
- What’s the expected number of 3’s in my number?

answers:

- $1/10$
- $1/10$
- $1/10$
- probability both last and second to last are 3 is $1/100$, so probabilities multiply and they are independent.
- $= 6 \cdot \frac{1}{10} = 3/5$

I have a class with 10 students.

1. How many different ways can I line them up, first to tenth?
2. If I pick a random way of lining them up, what’s the probability the shortest student is in front?
3. What’s the probability the tallest student is in back?
4. Are these events independent?

- $10!$
- $1/10$
- $1/10$
- probability shortest in front and tallest in back $= \frac{1}{10} \cdot \frac{1}{5} \neq \frac{1}{10} \cdot 110$ so they are not independent.
In no more than 10 sentences, describe how you would use statistics to answer the following question: “Does higher income lead to better health?”

After describing your procedure for answering that question, critique it: what might be the potential problems or biases in your method?

A policeman’s radar gun says ‘speeding’ with probability .9 if a car is actually speeding. If the car is not speeding, it still will say ‘speeding’ with probability .01. 5% of drivers on the Garden Parkway are speeding. If The policeman points his radar gun at a random car and it says ‘speeding’ what’s the chance the car is actually speeding?

Recall the geometric distribution: $X$ is the number of tails before you get one head with a $p$-biased coin.

i.e. 

$$
Pr[X = i] = (1 - p)^i p, \text{ for } i = 0, 1, \ldots
$$

• Calculate $E[X]$

• Calculate $var(X)$

You do the following. You flip a fair coin until you get a head. You then continue flipping until you get a tail after that. (So you flip the coin at least twice, and exactly twice only if you flip H, T. Another possible sequence is TTTHHHHT). Let $X$ be the total number of flips, $Y$ the total number of heads, $Z$ the total number of tails.

• Calculate $E[X], E[Y], E[Z]$

Write the definition of two random variables $X$ and $Y$ being independent.

There is a lottery in which all 365 days of the year are put into a big hopper, and a random date drawn out. Whoever’s birthday it is wins the lottery, and that date returned to the hopper.

The lottery happens 40 times, and only 1 of the winning days is a December birthday. The December birthdays are angry and think the lottery is biased.

What do you say? Justify your answer.

Let $Y$ be the number of heads in $n$ flips of a $p$-biased coin.

• What is $E[Y]$?

• What is $var(Y)$?
Let $X$ be a random variable, and let $Y = 2 - X$.

If $\text{var}(X) = 10$, what is $\text{var}(Y)$?

answer: $\text{var}(Y) = 10$, not independent because $X$ is not constant and $\{X \leq t\}$ and $\{Y \geq 2 - t\}$ are the same event, so probabilities don't multiply.

Let $A$ be the event that I get all heads in 6 flips of a $p$-biased coin, and $B$ the event I get all tails.

1. Are $A$ and $B$ independent?
2. What is $\Pr[A|B^c]$?
3. What is $\Pr[A|B]$?
4. What is $\Pr[A \cap B]$?
5. What is $\Pr[A \cup B]$?

answers:

- no, disjoint $\Pr(A \cap B) = 0.$
- $\frac{p^6}{1-(1-p)^6}$
- 0
- 0
- $p^6 + (1 - p)^6$