1 Conditional Distributions

1.1

Let $X$ and $Y$ be independent Poisson random variables with mean 1. Conditioned on $X + Y = 2$, are $X$ and $Y$ independent? Prove it.

They are not independent. Here’s a quick way to prove it. Conditioned on $X + Y = 2$, there is some positive probability that $X = 0$ and some positive probability that $Y = 0$. But there is 0 probability that $\{X = 0 \land Y = 0\}$. So the random variables are not independent.

1.2

Let $X$ be an exponential random variable with mean 1 (density function $e^{-x}$).

What is $\Pr[X \geq 3 | X \geq 1]$?

\[
\Pr[X \geq 3 | X \geq 1] = \frac{\Pr[X \geq 3 \land X \geq 1]}{\Pr[X \geq 1]}
\]

\[
= \frac{\Pr[X \geq 3]}{\Pr[X \geq 1]}
\]

\[
= \frac{\int_3^\infty e^{-x} \, dx}{\int_1^\infty e^{-x} \, dx}
\]

\[
= \frac{e^{-3}}{e^{-1}} = e^{-2}
\]
2 Convergence of Random Variables

2.1 Let $U$ be a uniform $[0, 1]$ random variable. For $k \geq 1$, let $X_k = 1$ if $U \geq 1 - \frac{1}{k}$ and 0 otherwise.

- As $n \to \infty$ does $X_n \to 0$ in distribution? Prove it.
- Does $X_n \to 0$ in probability? Prove it.
- Does $X_n \to 0$ almost surely? Prove it.

1. $X_n \to 0$ in distribution. Many ways to prove it but here's one. $\phi_0(t) = e^{it} = 1$. $\phi_{X_n}(t) = \frac{1}{n} e^{it} + 1 - \frac{1}{n} \to 1$ as $n \to \infty$.

2. $X_n \to 0$ in probability. $\Pr[|X_n - 0| > \epsilon] = \frac{1}{n} \to 0$ as $n \to \infty$.

3. $X_n \to 0$ almost surely. With probability 1, $U < 1$. Therefore there is some $N$ large enough, that $1 - 1/N > U$. In that case $X_n = 0$ for all $n \geq N$, and so $\lim_{n \to \infty} X_n = 0$. Since this happens with probability 1, $X_n \to 0$ almost surely.

Another way to see this is the following: The sequence $X_1, X_2, X_3, \ldots$ will look like this: $1, 1, 1, 1, 1, 0, 0, 0, \ldots$ where $X_k = 1$ for all $k \leq \frac{1}{1-U}$ and 0 for all $k > \frac{1}{1-U}$. If $U \neq 1$, $\frac{1}{1-U}$ is finite, and the sequence therefor has the limit 0. $U \neq 1$ with probability 1, so $X_k \to 0$ almost surely.
3 Limit Theorems

3.1
Say $X_1, X_2, \ldots$ are iid with mean 0, variance 1. Prove that (without just quoting a theorem)
\[
\frac{X_1 + \cdots + X_n}{n} \to 0 \text{ in probability}
\]

Let $U_n = \frac{X_1 + \cdots + X_n}{n}$. By Chebyshev’s Inequality,
\[
\Pr[|U_n| > \epsilon] \leq \frac{\text{var}(U_n)}{\epsilon^2} = \frac{1}{n\epsilon^2} \to 0
\]

3.2
Let $X_n$ be Poisson with mean $n$. Prove that $\frac{X_n - n}{\sqrt{n}} \to N(0, 1)$ in distribution. You may quote a theorem as part of your proof (but you can prove this other ways as well).

Two possible ways to show this:

1. $X_n$ has the same distribution as $\sum_{i=1}^{n} Y_i$ where $Y_i$’s are independent Poisson random variables with mean 1. The Central Limit Theorem says that
\[
\frac{\sum_{i=1}^{n} (Y_i - 1)}{\sqrt{n}} \xrightarrow{D} N(0, 1)
\]

2. We can use characteristic functions:
Fact 1:
\[
\phi_{X_n}(t) = \sum_{k=0}^{\infty} \frac{e^{itn^k}e^{-n}}{k!} = e^{n(e^{it}-1)}
\]
so
\[
\phi_{(X_n-n)/\sqrt{n}}(t) = e^{-it\sqrt{n}}e^{n(e^{it/\sqrt{n}}-1)}
\]
Now to find the limit as $n \to \infty$, we use a power series expansion of the inner-most exponential,
\[
\phi_{(X_n-n)/\sqrt{n}}(t) = e^{-it\sqrt{n}}e^{n(it/\sqrt{n}-t^2/2n+o(1/n))} = e^{-t^2/2+o(1)}
\]
and since $e^{-t^2/2}$ is the characteristic function of a $N(0, 1)$ random variable, the proof is complete.