In many theorems and questions in probability theory, the perspective is *asymptotic*: there is some parameter $n$, and we are interested in characterizing behavior as $n$ gets very large. The famous theorems in probability have this perspective: the Law of Large Numbers and the Central Limit Theorem.

We need some notation and techniques to deal with asymptotics efficiently.
Asymptotic Equivalence

We write:

\[ f(n) \sim g(n) \]

if

\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \]

Examples:

1. \( n^2 - 100n + 27 \sim n^2 \)
2. \( \frac{n}{n - \log n} \sim 1 \)
3. \( {n \choose 7} \sim \frac{n^7}{7!} \)
Big-Oh Notation

We write:

\[ f(n) = O(g(n)) \]

if there is some \( K \) so that

\[ \limsup_{n \to \infty} \frac{f(n)}{g(n)} \leq K \]

In other words, there is some \( N \) so that for all \( n \geq N \),
\( f(n) \leq Kg(n) \).

Examples:

1. \( 10n^2 + 100n = O(n^2) \)
2. \( 100n = O(n^2) \)
3. \( \binom{n}{7} = O(n^7) \)
We write:

\[ f(n) = \Theta(g(n)) \]

if \( f(n) = O(g(n)) \) and \( g(n) = O(f(n)) \). I.e. there is some \( 0 < c, K < \infty \) so that

\[ c \leq \limsup_{n\to\infty} \frac{f(n)}{g(n)} \leq K \]

Examples:

1. \( 10n^2 + 100n = \Theta(n^2) \)
2. \( \binom{n}{7} = \Theta(n^7) \)
We write:

\[ f(n) = o(g(n)) \]

if

\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \]

In other words, for all \( \epsilon > 0 \), there is some \( N \) so that for all \( n \geq N \),

\[ f(n) \leq \epsilon g(n) \]

Examples:

1. \( 10n^2 + 100n = o(n^3) \)
2. \( 100n = o(n^2) \)
3. \( \left( \frac{n}{7} \right) = o(n^8) \)
We write:

\[ f(n) = \Omega(g(n)) \]

if there is some \( c > 0 \) so that for sufficiently large \( n \),

\[ f(n) \geq cg(n) \]

We write:

\[ f(n) = \omega(g(n)) \]

if

\[ \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0 \]

Exmaples:

1. \( \binom{n}{7} = \Omega(n^7) \)
2. \( \binom{n}{7} = \omega(n^6) \)
Often we are concerned with ‘typical’ behavior as $n \to \infty$.

One definition of a typical event is that the probability tends to 1 as $n \to \infty$. I.e. $\Pr(A) = 1 - o(1)$ in little-oh notation.

The following are all equivalent ways of saying the same thing:

1. $\Pr(A) \to 1$ as $n \to \infty$
2. $\Pr(A) = 1 - o(1)$
3. $A$ occurs ‘with high probability’ or ‘whp’.
Theorem (Stirling’s Formula)

\[ n! \sim n^n e^{-n} \sqrt{2\pi n} \]
There are a few power series that are helpful in finding asymptotics:

1. \( e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^k}{k!} + \cdots \)
2. \( \log(1 + x) = x - \frac{x}{2} + \frac{x}{3} - \cdots \)
3. \( \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \)
Useful Limits

\[(1 + \frac{x}{n})^n \to e^x\]
What is the probability that a simple symmetric random walk = 0 at step $n$? Assume $n$ is even. [Describe simple symmetric random walk]. This is the same as the probability a $Bin(n, 1/2) = 0$. 

Exact:

$$\Pr[S_n = 0] = \binom{n}{n/2} (1/2)^n$$
Example

Asymptotics: use Stirling’s Formula and cancel:

\[
\binom{n}{n/2} \left(\frac{1}{2}\right)^n = \frac{n!2^{-n}}{(n/2)!/(n/2)!}
\]

\[
\approx \frac{n^n e^{-n} \sqrt{2\pi n} 2^{-n}}{(n/2)^n e^{-n} \pi n}
\]

\[
\approx \sqrt{\frac{2}{\pi n}}
\]
For constant $k$, we know that $\binom{n}{k} \sim n^k / k!$. Does that still hold if $k$ depends on $n$?

For $k = k(n)$, find the asymptotics of:

\[
\frac{\binom{n}{k}}{n^k / k!}
\]

Find for:

- $k = o(n^{1/2})$
- $k = o(n^{2/3})$
All of the above definitions can be used with other parameters, besides $n \to \infty$. For example,

$$5x^2 + 3x \sim 3x$$

as $x \to 0$. 