Abstract Measures and Integration

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Recall that a $\sigma$-field $\mathcal{F}$ is a collection of subsets of a sample space $\Omega$ so that:

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. If $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcup A_i \in \mathcal{F}$
Definitions

\( \mu : \mathcal{F} \rightarrow [0, \infty) \) is a measure if:

- \( \mu(\emptyset) = 0 \)
Measures

**Definition**

$\mu : \mathcal{F} \to [0, \infty)$ is a *measure* if:

- $\mu(\emptyset) = 0$
- If $A_1, A_2, \ldots$ are disjoint, then

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$
A *probability measure* is simply a measure $\mu$ with $\mu(\Omega) = 1$. 
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**Exercise:** How can you characterize all probability measures on $(\mathbb{R}, \mathcal{B})$? Give several examples of different probability measures on $\mathbb{R}$. 
What is the Riemann Integral and why do we need a new one?
Small problem: some (strange) functions are not integrable. 
eg. Let \( f(x) = 1 \) if \( x \) is rational, 0 if \( x \) is irrational. Prove that \( f \) is not Riemann integrable.
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Big problem: Riemann integral has a problem with limits.
Riemann Integration

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Big problem: Riemann integral has a problem with limits. Pointwise limit of a sequence $f_n$ of Riemann integrable functions may not be integrable! (that’s a bad thing..)
Measurable Functions

**Definition**

A *measurable function* $f : \Omega \rightarrow \mathbb{R}$ on $(\Omega, \mathcal{F})$ is a function so that $f^{-1}(U) \in \mathcal{F}$ for every open set $U \subseteq \mathbb{R}$. 
Need to know some definitions from analysis:

- sup, inf, lim sup, lim inf
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In the HW: Prove that if $f_n$ is a sequence of measurable functions then $\lim\sup f_n, \lim\inf f_n, \sup f_n, \inf f_n$ are all measurable functions.
Definition

\(1_A(x)\) is the *indicator function* of a set \(A\) if \(1_A(x) = 1\) if \(x \in A\) and \(1_A(x) = 0\) if \(x \notin A\).
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When is an indicator function a measurable function?
Simple Functions

Definition

A simple function is a function whose range is a finite set. I.e.

\[ f(x) = \sum_{i=1}^{N} \alpha_i 1_{A_i} \]
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\( f(x) \) is measurable if \( A_i \in \mathcal{F} \) for all \( i \).
The Integral

Let \( f = \sum_{i=1}^{N} \alpha_i 1_{A_i} \) be a non-negative, measurable simple function. Then the integral of \( f \) with respect to the measure \( P \) is defined as:

\[
\int f = \sum_{i=1}^{N} \alpha_i P(A_i)
\]
Let $f$ be a non-negative measurable function. Then the integral of $f$ is defined as:

\[ \int f = \sup_{g \leq f, \text{simple}} \int g \]
The Integral

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Definition
A non-negative measurable function \( f \) is integrable if \( \int f < \infty \).
The Integral

Let \( f \) be a measurable function. Define \( f^+(x) = \max\{f(x), 0\} \), \( f^-(x) = \max\{-f(x), 0\} \).

Then \( f(x) = f^+(x) + f^-(x) \) and we define

\[
\int f = \int f^+ - \int f^-
\]

when the subtraction makes sense (i.e. at least one is \( < \infty \)).
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$$\int f = \int f^+ - \int f^-$$

when the subtraction makes sense (i.e. at least one is $< \infty$).

**Definition**

A measurable function $f$ is integrable if both $f^+$ and $f^-$ are integrable.
Properties of the Integral

1. If $f \geq 0$ a.e., then $\int f \geq 0$
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4. \( \int af = a \int f \)
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6. Jensen’s Inequality: If $g : \mathbb{R} \to \mathbb{R}$ is convex, then

$$g \left( \int f \right) \leq \int g(f)$$
How to prove that $\int (f + g) = \int f + \int g$?

Begin with simple functions. Let $f = \sum \alpha_i 1_{A_i}$ and $g = \sum \beta_j 1_{B_j}$.

Set $A_0 = \bigcup A_i$ and $B_0 = \bigcup B_j$.

Then $(f + g) = N \sum_{i,j} (\alpha_i + \beta_j) 1_{A_i \cap B_j}$

**Example with one set each**

Now use properties of a measure to show that $\int (f + g) = \int f + \int g$.

How to extend for general measurable functions?
How to prove that \( \int (f + g) = \int f + \int g \)?

Begin with simple functions. Let \( f = \sum_i \alpha_i 1_{A_i} \) and \( g = \sum_j \beta_j 1_{B_j} \).

Set \( A_0 = (\bigcup A_i)^c \) and \( B_0 = (\bigcup B_i)^c \).
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Then

$$(f + g) = \sum_{i,j=0}^N (\alpha_i + \beta_i) 1_{A_i \cap B_j}$$

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Now use properties of a measure to show that

$$\int (f + g) = \int f + \int g$$
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Then

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**Example with one set each**

Now use properties of a measure to show that

$$ \int (f + g) = \int f + \int g $$

How to extend for general measurable functions?
Monotone Convergence Theorem

Suppose \( f_n \geq 0, f_n(x) \to f(x) \) almost surely and \( f_n(x) \) is non-decreasing in \( n \). Then

\[
\lim_{n \to \infty} \int f_n = \int f
\]
Theorem

Suppose $f_n \geq 0$, $f_n(x) \rightarrow f(x)$ almost surely and $f_n(x)$ is non-decreasing in $n$. Then

$$\lim_{n \rightarrow \infty} \int f_n = \int f$$

Proof?
Fatou’s Lemma

Theorem

If \( f_n \geq 0 \), then

\[
\liminf_{n \to \infty} \int f_n \geq \int (\liminf_{n \to \infty} f_n)
\]
Theorem

Suppose $|f_n(x)| \leq g(x)$ for all $x$ and $n$, $f_n(x) \to f(x)$ a.s., and $\int g < \infty$. Then

$$\lim_{n \to \infty} \int f_n = \int f$$
Counterexamples

When does $\int f_n$ not converge to $\int f$?

Example 1: Let $f_n = n \cdot 1_{(0, 1/n]}$. Then $\int f_n = 1$ for all $n$, but $f_n(x) \to 0$ a.s.

Example 2: Let $f_n = 1_{[-n, n]}$. Then $\int f_n = 2$ for all $n$, but $f_n(x) \to 0$ a.s.
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Example 1: Let $f_n = n \cdot 1_{(0,1/n)}$. Then $\int f_n = 1$ for all $n$, but $f_n(x) \to 0$ a.s.

Example 2: Let $f_n = \frac{1}{n} 1_{[-n,n]}$. Then $\int f_n = 2$ for all $n$, but $f_n(x) \to 0$ a.s.