Green’s Theorem. Gauss’s Divergence Theorem.

[1] For a planar domain $G$ with boundary $\partial G$ and real-valued functions $p(x, y)$ and $q(x, y)$, Green’s theorem states that

$$\int_{\partial G} p(x, y)dx + q(x, y)dy = \iint_G \left( \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy,$$

under appropriate conditions on the smoothness of functions $p(x, y), q(x, y)$, and the boundary curve $\partial G$.

Prove Green’s theorem under the following conditions:

- The functions $p, q : \Omega \to \mathbb{R}$ are differentiable everywhere in a planar domain $\Omega$;
- The function $\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$ is continuous in $\Omega$; (we are not assuming that $p, q$ are $C^1$)
- Domain $G$ is the interior of a triangle with $\overline{G} \subset \Omega$.

[2] Assume that

- The planar domain $\Omega$ is simply connected.
- The functions $p, q : \Omega \to \mathbb{R}$ are differentiable everywhere in a planar domain $\Omega$ and satisfy

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$

Show that there exists a $C^1$ function $f : \Omega \to \mathbb{R}$ such that $\text{grad } f = (p, q)$.

[3] For a vector field $\mathbf{F}$ defined on a 3-D solid domain $G$ with boundary surface $\partial G$ and the unit outward normal $\mathbf{n}$, Gauss’s divergence theorem states that

$$\iint_{\partial G} \mathbf{F} \cdot \mathbf{n}dS = \iiint_G \text{div } \mathbf{F} dxdydz,$$
under appropriate conditions on the smoothness of \( \mathbf{F}(x, y, z) \), and the boundary surface \( \partial G \).

Prove Gauss’s divergence theorem under the following conditions:

- The vector field \( \mathbf{F} = (F_1, F_2, F_3) \) is defined on a 3-D solid domain \( \Omega \) such that each component \( F_i : \Omega \to \mathbb{R} \) is differentiable everywhere in \( \Omega \);
- The divergence \( \text{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \) is continuous in \( \Omega \); (we are not assuming that \( F_1, F_2, F_3 \) are \( C^1 \))
- Domain \( G \) is the interior of a tetrahedron such that \( \overline{G} \subset \Omega \).