Exercises for Taylor Series and Laurent Series

[1] Find the Taylor series of $f(z)$ expanded about the given point. Give the region where the series converges.

(a) $f(z) = 1/(z + 2)$ expanded about $z = 0$.
(b) $f(z) = 1/(z + 2)$ expanded about $z = 3i$.
(c) $f(z) = z^5/(z^3 - 4)$ expanded about $z = 0$.
(d) $f(z) = z \sin z$ expanded about $z = \pi/2$.
(e) $f(z) = \log z$ expanded about $z = 3$.

[2] Consider the Taylor series of $f(z) = 1/(2 - \sin z)$ about the origin $z = 0$. What is the radius of convergence of this Taylor series?

[3] Give the first three nonzero terms of the Taylor series of the given function about the specified point. Find the radius of convergence of the Taylor series.

(a) $f(z) = e^{3z-2}z^2$ about $z = 0$.
(b) $f(z) = \cot z = \frac{\cos z}{\sin z}$ about $z = \pi/2$.
(c) $f(z) = \log(i + e^{2z})$ about $z = 0$.

[4] Find the Laurent series of $f(z)$ expanded about the given point:

(a) $f(z) = \frac{1}{(1 - z)(z + 2)}$ expanded about $z = 0$ for $|z| < 1$
(b) $f(z) = \frac{1}{(1 - z)(z + 2)}$ expanded about $z = 0$ for $1 < |z| < 2$
(c) $f(z) = \frac{1}{(1 - z)(z + 2)}$ expanded about $z = 0$ for $|z| > 2$
(d) $f(z) = z^3 \cos(1/z)$ expanded about $z = 0$ for $|z| > 0$.
(e) $f(z) = \sin \left( \frac{z}{1 - z} \right)$ expanded about $z = 1$ for $|z - 1| > 0$.
(f) $f(z) = \log \left( 1 - \frac{2}{z^2} \right)$ expanded about $z = 0$ for $|z| > \sqrt{2}$.

See next page for the answers
ANSWERS

[1] (a) \( f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n \) for \(|z| < 2\)

(b) \( f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2 + 3i)^{n+1}} (z - 3i)^n \) for \(|z - 3i| < |2 - 3i| = \sqrt{13}\)

(c) \( f(z) = \sum_{n=0}^{\infty} -\frac{1}{4^{n+1}} z^{3n+5} \) for \(|z| < 4^{1/3}\)

(d) \( f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi/2}{(2n)!} (z - \pi/2)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z - \pi/2)^{2n+1} \) for \(|z| < \infty\)

(e) \( f(z) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 3^n} (z - 3)^n \) for \(|z - 3| < 3\)

[2] \( \sqrt{\frac{\pi^2}{4} + \left[ \ln \left(2 - \sqrt{3}\right) \right]^2} \)

Hint: It’s quite hopeless to solve the problem by finding the values of Taylor coefficients and then carefully studying their asymptotics as \( n \to \infty \). Rather we should examine: What is the largest disk about the origin in which \( f(z) \) is complex analytic?

[3] (a) \( 1 + 3z + \frac{7}{2}z^3 \), \( R = \infty \).
(b) \( -(z - \frac{\pi}{2}) - \frac{1}{3}(z - \frac{\pi}{2})^3 - \frac{2}{15}(z - \frac{\pi}{2})^5 \), \( R = \pi/2 \).
(c) \( \left(\frac{1}{2} \ln 2 + i \frac{\pi}{4}\right) + (1 - i)z + z^2 \), \( R = \pi/4 \).

[4] (a) \( f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{3} + \frac{(-1)^n}{6 \cdot 2^n}\right) z^n \) for \(|z| < 1\)

(b) \( f(z) = \sum_{n=0}^{\infty} \frac{-1}{3} z^{-1-n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{6 \cdot 2^n} z^n \) for \( 1 < |z| < 2 \)

(c) \( f(z) = \sum_{n=0}^{\infty} \left(\frac{-1}{3} + \frac{(-2)^n}{3}\right) z^{-1-n} \) for \( 2 < |z| < \infty \)

(d) \( f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{3-2n} \) for \( 0 < |z| < \infty \)

(e) \( f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \sin 1}{(2n)!} (z - 1)^{-2n} + \sum_{n=0}^{\infty} \frac{(-1)^n \cos 1}{(2n+1)!(z - 1)^{-2n-1}} \) for \( 0 < \left|z - 1\right| < \infty \)

(f) \( f(z) = -\sum_{n=1}^{\infty} \frac{2^n}{n} z^{-2n} \) for \( |z| > \sqrt{2} \)