Exercises for Stereographic Projection

Let $S$ be the Riemann sphere:

$$X^2 + Y^2 + (Z - 1/2)^2 = 1/4,$$

with the North Pole $N = (0, 0, 1)$.
Let $\Phi : S \setminus \{N\} \to \mathbb{C}$ be the stereographic projection.

1. Find the explicit formulas for $\Phi$ and $\Phi^{-1}$.

2. Two points on the Riemann sphere are antipodal if and only if their stereographic projections $z_1$ and $z_2$ satisfy $z_1 \overline{z_2} = -1$.

3. What is the image of the lower hemisphere under the stereographic projection? What about the upper hemisphere?

4. (a) Show that $C$ is a circle on $\mathbb{C}$ if and only if $\Phi^{-1}(C)$ is a circle on $S$ that does not pass through the North Pole $N$.
   (b) Show that $L$ is a straight line on $\mathbb{C}$ if and only if $\Phi^{-1}(L) \cup \{N\}$ is a circle on $S$ that passes through $N$.
   (c) What are the stereographic projections of great circles on $S$ that pass through $N$?

5. The stereographic projection is conformal; in other words, the transformation does not change the intersecting angle of curves.

More precisely, let $C_1$ and $C_2$ be two smooth curves in $\mathbb{C}$ intersecting at a point $z$ with angle $\theta$ (that means the angle between the tangent lines of the two curves at $z$ is $\theta$). Show that the angle between $\Phi^{-1}(C_1)$ and $\Phi^{-1}(C_2)$ at $\Phi^{-1}(z)$ is also equal to $\theta$. 