The Riemann Zeta Function
Part 2: pole and zeros

THEOREM (Pole and Trivial Zeros of \( \zeta(s) \)):

(a) \( \zeta(s) \) is holomorphic in \( \mathbb{C} \setminus \{1\} \) and has a simple pole at \( s = 1 \) with residue \( 1 \); in other words, \( \zeta(s) = \frac{1}{s-1} + (\text{an entire function}) \).

(b) The only zeros of \( \zeta(s) \) with \( \Re s \leq 0 \) or \( \Re s \geq 1 \) are the simple zeros at negative even integers \( s = -2, -4, \cdots \).

The mail tools of the proof are the Euler product and Riemann’s functional equation.

**Euler Product:** \( \zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \) (\( \Re s > 1 \)).

**Riemann’s Functional Equation:** \( \zeta(s) \equiv 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s)\zeta(1-s) \).

**Outline of a Proof of the Theorem**

(i) The only pole of \( \zeta(s) \) is at \( s = 1 \), where the residue = 1.
   Keys: Evaluate \( G(1) \) directly by a residue calculation, plus some basic properties of the Gamma function.

(ii) \( \zeta(s) \neq 0 \) for \( \Re s > 1 \).
   Keys: The Euler product.

(iii) The only zeros of \( \zeta(s) \) in \( \Re s < 0 \) are the simple zeros at negative even integers.
   Keys: (ii) and Riemann’s functional equation in addition to the basics of the Gamma function. (Lemma 1 gives an alternative method of proving the vanishing of \( \zeta(-2n) \) by evaluating \( G(-2n) \) directly.)

(iv) \( \zeta(s) \) has no zeros on the vertical line \( \Re s = 1 \).
   Keys: The log derivative of the Euler product and an elementary trick (Hadamard-de la Vallée Poussin).

(v) \( \zeta(s) \) has no zeros on the vertical line \( \Re s = 0 \).
   Keys: This follows from (i), (iv), and Riemann’s functional equation immediately.

The proof of the Euler product is relatively easy. Simply expand \( (1 - p^{-s})^{-1} \) into \( \sum_{k=0}^{\infty} p^{-ks} \) and multiply these series together.

Next in this note we will prove (i)(ii)(iii). A proof of Riemann’s functional equation based on the residue analysis will be given at the end of this note.

Step (iv) will be done later in Part 3.
LEMMA 1: (a) \( G(n) = 0 \) for integer \( n \geq 2 \).
(b) \( G(1) = 2\pi i \).
(c) \( G(-n) = 2\pi i \frac{a_{n+1}}{(n+1)!} \) for integers \( n \geq 0 \). Here \( a_k \) are the Taylor coefficients in
\[
\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k.
\]
In particular, \( G(-2) = G(-4) = \cdots = 0 \).
(d) The only pole of \( \zeta(s) \) is a simple pole at \( s = 1 \) with residue \( 1 \).
(e) \( \zeta(-2) = \zeta(-4) = \cdots = 0 \).

Proof: We have shown in Part 1 that:
\[
G(s) = \int_{C(\delta)} \frac{z^{s-1}}{e^z - 1} dz = \int_{|z|=\delta} \frac{z^{s-1}}{e^z - 1} dz + (e^{i2\pi s} - 1) \int_{\delta}^{\infty} \frac{x^{s-1}}{e^x - 1} dx,
\]
where the integration contour \( C(\delta) \) with \( 0 < \delta < 2\pi \) is:

(Recall also that \( G(s) \) is independent of \( \delta \).)

Proof of (a)(b)(c): Let \( s \) in the above formula be an integer: \( s = n \). The second integral on the right hand side vanishes, and thus
\[
G(n) = \int_{|z|=\delta} \frac{z^{n-1}}{e^z - 1} dz = 2\pi i \text{Res} \left( \frac{z^{n-1}}{e^z - 1}; 0 \right).
\]
The rest is easy. Let’s just remark that the function
\[
\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z e^{z/2} + e^{-z/2}}{2 e^{z/2} - e^{-z/2}}
\]
is an even function. This implies
\[
a_1 = -\frac{1}{2}, a_3 = a_5 = a_7 = \cdots = 0.
\]
The first few \( a_{2k} \) are
\[
a_0 = 1, a_2 = \frac{1}{6}, a_4 = -\frac{1}{30}, a_6 = \frac{1}{42}, a_8 = -\frac{1}{30}, a_{10} = \frac{5}{66}, \cdots,
\]
and they are connected to the so-called Bernoulli numbers: \( a_{2k} = (-1)^{k-1} B_{2k} \).

Proof of (d)(e): Apply the results in (a)(b)(c) to
\[
\zeta(s) = \frac{G(s)}{(e^{i2\pi s} - 1)\Gamma(s)} \quad (s \in \mathbb{C}).
\]
From this definition of \(\zeta(s)\), we see that the only possible singularities of \(\zeta(s)\) are the zeros of \(e^{i2\pi s} - 1\), that is, \(s = \text{integers}\). Recall that for positive integers \(n \geq 1\), \(\Gamma(n) = (n-1)!\), and for nonpositive integers \(s = -n \leq 0\), \(\Gamma(s)\) has a simple pole at \(s = -n\) with residue \(\text{Res}(\Gamma; -n) = (-1)^n/n!\). The statements (d)(e) will follow readily.

**Lemma 2:** \(\zeta(s) \neq 0\) for \(\text{Re}\,s > 1\).

*Proof:* We use the Euler product formula.

Write \(s = \sigma + it\) with \(\sigma > 1\) and \(t \in \mathbb{R}\). Recall that for all real \(x\),

\[
e^x - (1 + x) = \int_0^x (x - u)e^u du \geq 0.
\]

(This is the integral form of the remainder for the Taylor approximation \(e^x \approx 1 + x\).) We have

\[
|1 - p^{-s}| \leq 1 + |p^{-s}| = 1 + p^{-\sigma} \leq \exp (p^{-\sigma}).
\]

Use this in the Euler product:

\[
|\zeta(s)| = \prod_{p:\text{prime}} |1 - p^{-s}|^{-1}
\]

\[
\geq \prod_{p:\text{prime}} \exp (-p^{-\sigma}) = \exp \left( - \sum_{p:\text{prime}} p^{-\sigma} \right)
\]

\[
\geq \exp \left( - \sum_{n=2}^{\infty} n^{-\sigma} \right) > 0.
\]

**Lemma 3:** The only zeros of \(\zeta(s)\) in \(\text{Re}\,s < 0\) are the simple zeros at negative even integers \(s = -2, -4, \ldots\)

*Proof:* Let \(\text{Re}\,s < 0\). We are going to use Riemann’s functional equation:

\[
\zeta(s) \equiv 2^s\pi^{-s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).
\]

First, among the factors on the right hand side, \(\zeta(1-s) \neq 0\) by Lemma 2. Moreover, \(\zeta(1-s)\) is holomorphic in \(\text{Re}\,s < 0\).

Second, \(\Gamma(z)\) has no zeros in \(\mathbb{C}\), since by definition

\[
\Gamma(z) = \frac{1}{\text{an entire function}}.
\]

We also know that \(\Gamma(z)\) has simple poles only at nonpositive integers. Thus, \(\Gamma(1-s) \neq 0\) and is holomorphic in \(\text{Re}\,s < 0\).

Thus, the zeros of \(\zeta(s)\) with \(\text{Re}\,s < 0\) are exactly the zeros of \(\sin(\pi s/2)\) with \(\text{Re}\,s < 0\), that is, \(s = -2, -4, \ldots\).

Moreover, differentiating the functional equation at \(s = -2n\), we obtain

\[
\zeta'(-2n) = (-1)^n n^{-2n-1} \pi^{-2n} \Gamma(2n+1) \zeta(2n+1) \neq 0.
\]
Therefore, negative even integers $s = -2n$ are simple zeros of $\zeta(s)$.

Now we give

**Proof of Riemann’s Functional Equation:** It suffices to prove the functional equation for $\operatorname{Re} s < 0$.

Take $0 < \varepsilon < \delta < 2\pi$ and a (large) positive integer $N$. Integrate $g(s, z) = z^{s-1}/(e^z - 1)$ on the following closed contour $C(\delta, \varepsilon, N)$ in the $z$-plane:

which consists of the black and red paths in the above figure.

Notice that the poles of $g(s, z)$ (as a function of $z$) enclosed by the closed contour $C(\delta, \varepsilon, N)$ are $2ni$ with $n = \pm 1, \pm 2, \cdots, \pm N$. By the residue theorem,

$$
\int_{C(\delta, \varepsilon, N)} g(s, z) dz = -2\pi i \sum_{-N \leq n \leq N, n \neq 0} \operatorname{Res} \left( \frac{e^{(s-1) \log z}}{e^z - 1}; z = 2n\pi i \right)
$$

(the negative sign in front of $2\pi i$ is due to the orientation of the contour)

$$
= -2\pi i \left[ \sum_{n=1}^{N} e^{(s-1)\ln(2n\pi) + i\pi/2} + \sum_{n=1}^{N} e^{(s-1)\ln(2n\pi) + i3\pi/2} \right]
$$

$$
= -2\pi i \sum_{n=1}^{N} [(2n\pi)^{s-1}e^{i(s-1)\pi/2} + (2n\pi)^{s-1}e^{i3(s-1)\pi/2}]
$$

$$
= -i(2\pi)^s \left[ e^{i(s-1)\pi/2} + e^{i3(s-1)\pi/2} \right] \sum_{n=1}^{N} n^{s-1}
$$

$$
\rightarrow 2i(2\pi)^s e^{i\pi s} \sin(\pi s/2)\zeta(1-s) \quad \text{as } N \rightarrow \infty.
$$
On the other hand, taking the limit as $N \to \infty$, we have
\[
\int_{\text{black}} g(s, z)dz \to \int_{C(\delta, \varepsilon)} g(s, z)dz = G(s) = (e^{2\pi i s} - 1)\Gamma(s)\zeta(s),
\]
\[
\int_{\text{red}} g(s, z)dz \to 0.
\]
The proof of the latter is based on the following uniform estimate on the red path:
\[
\max_{|z|=(2N+1)\pi} \frac{1}{|e^z - 1|} \leq M,
\]
where $M > 0$ is a positive constant independent of $N$. Using this we have
\[
\left| \int \text{red} g(s, z)dz \right| \leq [(2N + 1)\pi]^\text{Re}s M \int_0^{2\pi} e^{-\text{Im}s\theta}d\theta.
\]
When $\text{Re}s < 0$, the right hand side decays to 0 as $N \to \infty$. Passing to the limit, we obtain
\[
(e^{2\pi i s} - 1)\Gamma(s)\zeta(s) = 2i(2\pi)^s e^{i\pi s} \sin(\pi s/2)\zeta(1 - s),
\]
and hence
\[
\zeta(s) = \frac{2i(2\pi)^s e^{i\pi s} \sin(\pi s/2)\zeta(1 - s)}{(e^{2\pi i s} - 1)\Gamma(s)} = (2\pi)^s\zeta(1 - s) \frac{\sin(\pi s/2)}{\sin(\pi s)\Gamma(s)}.
\]
Recalling an indenity for the Gamma function:
\[
\Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin(\pi s)},
\]
we finally obtain
\[
\zeta(s) = (2\pi)^s\zeta(1 - s) \frac{\sin(\pi s/2)\Gamma(1 - s)}{\pi} \quad (\text{Re}s < 0).
\]
The proof is complete. 

**EXERCISES**

1. Show that $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ for $\text{Re}s > 1$. where $\mu(n)$ is defined as follows:
\[
\mu(n) = \begin{cases} 
1 & n = 1, \\
(-1)^k & n = p_1 \cdots p_k \text{ with distinct primes } p_1, \cdots, p_k, \\
0 & \text{otherwise}.
\end{cases}
\]
2. Evaluate $\zeta(0)$ and $\zeta'(0)$.
3. For positive integer $N$, let $M_N$ be
\[
M_N = \max_{|z|=(2N+1)\pi} \frac{1}{|e^z - 1|}.
\]
Show that $\limsup_{N \to \infty} M_N < \infty$. (This estimate was used in our proof of Riemann’s functional equation.)