Full Rank Matrix. Inverse Matrix

Rank and Nullity:

\[
\text{rank}(A) = \dim(\text{Range of } A) = \dim(\text{Column Space of } A) = \dim(\text{Row Space of } A) = \# \text{ of pivots in the echelon form of } A = \# \text{ of nonzero rows in the echelon form of } A = \text{the maximal number of linearly independent columns in } A = \text{the maximal number of linearly independent rows in } A.
\]

\[
\text{nullity}(A) = \dim(\text{Nullspace of } A).
\]

- If \( A \) is an \( m \times n \) matrix, then \( \text{rank}(A) + \text{nullity}(A) = n \).

**DEFINITION:** Let \( A \) be a square matrix of size \( n \).

An \( n \times n \) matrix \( B \) is called the inverse matrix of \( A \) if it satisfies

\[
AB = BA = I_n.
\]

The inverse of \( A \) is denoted by \( A^{-1} \).

If \( A \) has an inverse, \( A \) is said to be invertible or nonsingular.

If \( A \) has no inverses, it is said to be not invertible or singular.

**HOW TO COMPUTE?**

Row reduce \( \left[ \begin{array}{c|c} A & I_n \end{array} \right] \).

Case 1: The row echelon form becomes \( \left[ \begin{array}{c|c} I_n & B \end{array} \right] \).

In this case, the matrix \( B \) on the right half equals \( A^{-1} \).

Case 2: The left half has one entire row equal to zero.

In this case, the matrix \( A \) is not invertible.
EXERCISES:

[1] Suppose that a $4 \times 6$ matrix $A$ has rank 3.
   (a) Find the nullity of $A$.
   (b) The range of $A$ is
       \begin{align*}
       (i) & \ 0 \\
       (ii) & \ \mathbb{R}^4 \\
       (iii) & \ \mathbb{R}^6 \\
       (iv) & \ \text{none of the above.}
       \end{align*}
   (c) Does $A\mathbf{x} = 0$ have no solution, infinitely many solutions, or one solution?
   (d) True or False? $A\mathbf{x} = \mathbf{b}$ is always solvable for any vector $\mathbf{b}$ in $\mathbb{R}^4$.
   (e) True or False? $A\mathbf{x} = \mathbf{b}$ has at most one solution.
   (f) True or False? The columns of $A$ are linearly independent.

[2] Suppose that a $4 \times 6$ matrix $A$ has rank 4.
   (a) Find the nullity of $A$.
   (b) The range of $A$ is
       \begin{align*}
       (i) & \ 0 \\
       (ii) & \ \mathbb{R}^4 \\
       (iii) & \ \mathbb{R}^6 \\
       (iv) & \ \text{none of the above.}
       \end{align*}
   (c) Does $A\mathbf{x} = 0$ have no solution, infinitely many solutions, or one solution?
   (d) True or False? $A\mathbf{x} = \mathbf{b}$ is always solvable for any vector $\mathbf{b}$ in $\mathbb{R}^4$.
   (e) True or False? $A\mathbf{x} = \mathbf{b}$ has at most one solution.
   (f) True or False? The columns of $A$ are linearly independent.

[3] Suppose that $A$ is a $6 \times 4$ matrix such that $A\mathbf{x} = 0$ has only one solution $\mathbf{x} = 0$.
   (a) Find the nullity and rank of $A$.
   (b) The range of $A$ is
       \begin{align*}
       (i) & \ 0 \\
       (ii) & \ \mathbb{R}^4 \\
       (iii) & \ \mathbb{R}^6 \\
       (iv) & \ \text{none of the above.}
       \end{align*}
   (c) True or False? $A\mathbf{x} = \mathbf{b}$ is always solvable for any vector $\mathbf{b}$ in $\mathbb{R}^6$.
   (d) True or False? $A\mathbf{x} = \mathbf{b}$ has at most one solution.
   (e) True or False? The columns of $A$ are linearly independent.

[4] Suppose that a $6 \times 6$ matrix $A$ has rank 6.
   (a) Find the nullity of $A$.
   (b) The range of $A$ is
       \begin{align*}
       (i) & \ 0 \\
       (ii) & \ \mathbb{R}^6 \\
       (iii) & \ \mathbb{R}^{36} \\
       (iv) & \ \text{none of the above.}
       \end{align*}
(c) Does $A\vec{x} = 0$ have no solution, infinitely many solutions, or one solution?

(d) True or False? $A\vec{x} = \vec{b}$ is always solvable for any vector $\vec{b}$ in $\mathbb{R}^6$.

(e) True or False? $A\vec{x} = \vec{b}$ has at most one solution.

(f) True or False? The columns of $A$ are linearly independent.

(g) True or False? Matrix $A$ is invertible.

[5] For each given matrix, determine whether the matrix is invertible. If it is invertible, find its inverse matrix.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,  
(b) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$,  
(c) $\begin{bmatrix} 1 & 2 & -4 \\ 3 & 1 & 3 \\ 7 & -1 & 17 \end{bmatrix}$,  
(d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}$,

(e) $\begin{bmatrix} -1 & 1 & -1 & 2 \\ 1 & -2 & 3 & -3 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 2 & -2 \end{bmatrix}$,  
(f) $\begin{bmatrix} 0 & 1 & 2 & 4 \\ 2 & 1 & 1 & 3 \\ 1 & 3 & 4 & 10 \\ 0 & 0 & 2 & 2 \end{bmatrix}$.

[6] Given the fact that matrix $A = \begin{bmatrix} -1 & 1 & -1 & 2 \\ 1 & -2 & 3 & -3 \\ -1 & 1 & 2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$ has inverse $A^{-1} = \begin{bmatrix} 3 & -1 & -1 & 4 \\ -3 & -1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 4 & 0 & -1 & 3 \end{bmatrix}$, solve the equation $A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, by carrying out a matrix multiplication.

Turn over for the answers
Answers:

[1] (a) 3  (b) iv  (c) Infinitely many solutions  (d) False  (e) False  (f) False

[2] (a) 2  (b) ii  (c) Infinitely many solutions  (d) True  (e) False  (f) False

[3] (a) Nullity=0, Rank=4  (b) iv  (c) False  (d) True  (e) True

[4] (a) 0  (b) ii  (c) One solution $\vec{x} = 0$  (d) True  (e) True  (f) True  (g) True

[5] (a) $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

(b) $A$ is singular

(c) $A$ is singular

(d) $A^{-1} = \begin{bmatrix} 1 & -3/2 & 1/2 \\ 1 & -1 & 0 \\ -2 & 7/2 & -1/2 \end{bmatrix}$

(e) $A^{-1} = \begin{bmatrix} -1 & 1 & 1 & -2 \\ -2 & 0 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 \end{bmatrix}$

(f) $A$ is singular

[6] $\vec{x} = \begin{bmatrix} 0 \\ -5 \\ 1 \\ 4 \end{bmatrix}$