Null space and range of a matrix. Systems of linear equations. (continued)

1. True or False.

(a) The null space of a $4 \times 6$ real matrix is a subspace of $\mathbb{R}^6$.
(b) The column space of a $4 \times 6$ real matrix is a subspace of $\mathbb{R}^6$.
(c) The rank of a $6 \times 4$ real matrix is at most 4.
(d) The nullity of a $4 \times 6$ real matrix is at least 4.
(e) A vector $\tilde{b}$ in $\mathbb{R}^m$ is in the column space of an $m \times n$ real matrix $A$ if the linear system $A\tilde{x} = \tilde{b}$ is solvable.
(f) Any vector in the range of an $m \times n$ real matrix $A$ can be written as a linear combination of the column vectors of $A$.
(g) Suppose that an $m \times n$ real matrix $A$ is row reduced to another matrix $B$ by row reductions. Then the null space of $A$ equals the null space of $B$.
(h) Suppose that an $m \times n$ real matrix $A$ is row reduced to another matrix $B$ by row reductions. Then the range of $A$ equals the range of $B$.
(i) Suppose that an $m \times n$ real matrix $A$ is row reduced to another matrix $B$ by row reductions. Then $\text{rank}(A) = \text{rank}(B)$ and $\text{nullity}(A) = \text{nullity}(B)$.
(j) If the column vectors of a $6 \times 4$ real matrix $A$ are linearly independent, then the null space of $A$ is $\{0\}$.
(k) If the column vectors of a $6 \times 4$ real matrix $A$ are linearly independent, then the column space of $A$ is $\mathbb{R}^6$.

2. Let $\tilde{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix}$, $\tilde{u}_2 = \begin{bmatrix} 5 \\ -10 \\ 5 \\ 15 \end{bmatrix}$, $\tilde{u}_3 = \begin{bmatrix} -4 \\ 10 \\ -6 \\ -10 \end{bmatrix}$, $\tilde{u}_4 = \begin{bmatrix} 2 \\ -3 \\ 2 \\ 8 \end{bmatrix}$, $\tilde{u}_5 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find the dimension and a basis of $\text{Span}\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4, \tilde{u}_5\}$.
(b) Find the dimension and a basis of $\text{Span}\{\tilde{u}_1\}$. Is $\tilde{u}_2$ in $\text{Span}\{\tilde{u}_1\}$? If yes, write it as a scalar multiple of $\tilde{u}_1$ with a specific coefficient.
(c) Find the dimension and a basis of $\text{Span}\{\tilde{u}_1, \tilde{u}_2\}$. Is $\tilde{u}_3$ in $\text{Span}\{\tilde{u}_1, \tilde{u}_2\}$? If yes, write it as a linear combination of $\tilde{u}_1, \tilde{u}_2$ with specific coefficients.
(d) Find the dimension and a basis of $\text{Span}\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$. Is $\tilde{u}_4$ in $\text{Span}\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$? If yes, write it as a linear combination of $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ with specific coefficients.
(e) Find the dimension and a basis of $\text{Span}\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4\}$ Is $\tilde{u}_5$ in $\text{Span}\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4\}$? If yes, write it as a linear combination of $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4$ with specific coefficients.

Hint: The row reduction of one single matrix provides the answers to all these questions.
3. A $n \times n$ square matrix is called a *magic square* if its $n$ row sums, $n$ column sums, and 2 diagonal sums are all equal. For instance, \[
\begin{bmatrix}
8 & 1 & 6 \\
3 & 5 & 7 \\
4 & 9 & 2
\end{bmatrix}
\] is a $3 \times 3$ magic square, since
\[8 + 1 + 6 = 3 + 5 + 7 = 4 + 9 + 2 = 8 + 3 + 4 = 1 + 5 + 9 = 6 + 7 + 2 = 8 + 5 + 2 = 6 + 5 + 4 \]
\[
\begin{bmatrix}
1 & 1 & 4 \\
5 & 2 & -1 \\
0 & 3 & 3
\end{bmatrix}
\] is another one.

Let $V$ be the vector space of all $3 \times 3$ real matrices, and let $M$ be the set of all $3 \times 3$ magic squares with real entries.

(a) Show that $M$ is a subspace of $V$.

(b) Find the dimension and a basis of $M$.

See next page for the answers
Answers:

1. (a) Y  (b) N  (c) Y  (d) N  (e) Y  (f) Y  (g) Y  (h) N  (i) Y  (j) Y  (k) N

2. (a) \( \dim = 3 \). Basis: \( \vec{u}_1, \vec{u}_3, \vec{u}_4 \).
   (b) \( \dim = 1 \). Basis: \( \vec{u}_1 \). Yes, \( \vec{u}_2 = 5\vec{u}_1 \).
   (c) \( \dim = 1 \). Basis: \( \vec{u}_1 \). No.
   (d) \( \dim = 2 \). Basis: \( \vec{u}_1, \vec{u}_3 \). No.
   (e) \( \dim = 3 \). Basis: \( \vec{u}_1, \vec{u}_3, \vec{u}_4 \). Yes, \( \vec{u}_5 = 9\vec{u}_1 + \vec{u}_3 - 2\vec{u}_4 \).

3. (a) Verify by yourself. I’ll skip here.
   (b) \( \dim(M) = 3 \). The following matrices form a basis of \( M \):

\[
A_1 = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -2/3 & 1/3 & 4/3 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 4/3 & 1/3 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Remark 1: Of course you may have different choices. That’s just fine, as long as your basis consists of 3 different magic squares and they span \( M \).

Remark 2: Solving this problem allows us to generate all 3 \( \times \) 3 magic squares. For the example matrices given in the problem we have the following:

\[
\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} = 4A_1 + 9A_2 + 2A_3, \quad \begin{bmatrix} 1 & 1 & 4 \\ 5 & 2 & -1 \\ 0 & 3 & 3 \end{bmatrix} = 3A_2 + 3A_3.
\]