Vector Spaces and Subspaces

DEFINITION (Vector Space): A vector space is a nonempty set $V$ of “vectors” such that the vector addition and multiplication by real scalars are defined. First of all, the addition and multiplication must give vectors that are within $V$. And they need to satisfy the following 8 rules:

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. There is a “zero vector” $0$ such that $0 + u = u$ for all $u$
4. For any $u$ in $V$, there is a vector $-u$ such that $u + (-u) = 0$
5. $1u = u$
6. $(ab)u = a(bu)$
7. $a(u + v) = au + av$
8. $(a + b)u = au + bu$

DEFINITION (Subspace): Let $V$ be a vector space. A subspace of $V$ is a nonempty subset $S$ of $V$ satisfying the following two properties:

1. If $u$ and $v$ are in $S$, then $u + v$ is in $S$
2. If $u$ is in $S$ and $a$ is a scalar, then $au$ is in $S$
EXAMPLES: The following are vector spaces:

- \( \mathbb{R} \) (the set of all real numbers)
- \( \mathbb{R}^2 \) (the space of all vectors of the form \( \begin{bmatrix} x \\ y \end{bmatrix} \) with real numbers \( x \) and \( y \))
- \( \mathbb{R}^3 \) (the space of all vectors of the form \( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) with real numbers \( x_1, x_2, \) and \( x_3 \))
- \( \mathbb{R}^4 \) (the space of all vectors of the form \( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \) with real numbers \( x_1, x_2, x_3, \) and \( x_4 \))
- \( \mathbb{R}^n \) (the space of all vectors of the form \( \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) with real numbers \( x_1, x_2, \cdots, x_n \))
- The space of all real valued functions defined on the interval \([0,1]\)
- \( C[0,1] = \) the space of all real valued continuous functions defined on the interval \([0,1]\)
- \( C^1(-2,2) = \) the space of all real valued \( C^1 \) (i.e., continuously differentiable) functions defined on the interval \((-2,2)\)
- The space of all real valued continuous functions \( f \) defined on the interval \([-2,2]\) such that \( f(-2) = f(2) = 0 \)
- The space of all real valued continuous functions \( f \) defined on the interval \([0,1]\) such that \( \int_{0}^{1} f(x)dx = 0 \)
- The set of all polynomials of a single real variable with real coefficients
- The set of all sequences of real numbers
**EXAMPLES:** In each of the following, $S$ is a subspace of vector space $V$:

- $V = $ any vector space, $S = \{0\}$
- $V = $ any vector space, $S = V$
- $V = \mathbb{R}^2$, $S$ = a line in $\mathbb{R}^2$ through the origin
- $V = \mathbb{R}^3$, $S$ = a line in $\mathbb{R}^3$ through the origin
- $V = \mathbb{R}^3$, $S$ = a plane in $\mathbb{R}^3$ through the origin

- $V = \mathbb{R}^4$, $S$ = the space of all vectors of the form $\begin{bmatrix} 2t_1 \\ t_1 \\ t_2 \\ 3t_1 - t_2 \end{bmatrix}$ with reals $t_1$ and $t_2$
- $V = \mathbb{R}^4$, $S$ = the space of all vectors in $\mathbb{R}^4$ with the zero coordiantes $x_3 = x_4 = 0$
- $V = \mathbb{R}^5$, $S$ = the space of all vectors in $\mathbb{R}^5$ satisfying $2x_1 - 3x_2 + x_4 = 0, x_3 - 2x_5 = 0$
- $V = \mathbb{R}^n$, $S$ = the space of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ satisfying $x_1 + x_2 + x_3 + \cdots + x_n = 0$

- $V = $ the space of all real valued functions defined on the interval $[0, 1]$,
  
  \[ S = C[0, 1] = \text{the space of all real valued continuous functions defined on the interval } [0, 1] \]

- $V = $ the space of all real valued functions defined on the interval $[0, 1]$,
  
  \[ S = C^1[0, 1] = \text{the space of all real valued } C^1 \text{ functions defined on the interval } [0, 1] \]

- $V = C[0, 1], S = C^1[0, 1]$
- $V = C[-2, 2], S = \text{the space of all real valued continuous functions } f \text{ defined on the interval } [-2, 2] \text{ such that } f(-2) = f(2) = 0$
- $V = C[0, 1], S = \text{the space of all functions } f \text{ in } C[0, 1] \text{ satisfying } \int_0^1 f(x)dx = 0$
- $V = C[0, 1], S = \text{the set of all polynomials of a single real variable with real coefficients}$
- $V = C[0, 1], S = \text{the set of all polynomials of a single real variable with real coefficients with degree at most } 2$
- $V = $ the set of all sequences of real numbers,
  
  \[ S = \text{the set of all sequences of real numbers } \{a_n\} \text{ such that } \lim_{n \to \infty} a_n = 0 \]
EXAMPLES: In each of the following, $S$ is NOT a subspace of vector space $V$:

- $V = \mathbb{R}^2$, $S$ = a line in $\mathbb{R}^2$ not through the origin
- $V = \mathbb{R}^2$, $S$ = the first quadrant
- $V = \mathbb{R}^2$, $S$ = all vectors in $\mathbb{R}^2$ with integer coordinates
- $V = \mathbb{R}^2$, $S$ = the graph of a parabola
- $V = \mathbb{R}^2$, $S$ = a circle
- $V = \mathbb{R}^3$, $S$ = a line in $\mathbb{R}^3$ not through the origin
- $V = \mathbb{R}^3$, $S$ = a plane in $\mathbb{R}^3$ not through the origin
- $V = \mathbb{R}^3$, $S$ = a cylinder
- $V = \mathbb{R}^3$, $S$ = a rectangular box
- $V = \mathbb{R}^4$, $S$ = the space of all vectors in $\mathbb{R}^4$ with $x_1 = 3$
  \[
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
  \end{bmatrix}
  \]
- $V = \mathbb{R}^5$, $S$ = the space of all vectors in $\mathbb{R}^5$ satisfying $2x_1 - 3x_2 + x_4 = 0$, $x_3 - 2x_5 = 4$
- $V = \mathbb{R}^n$, $S$ = the space of all vectors in $\mathbb{R}^n$ satisfying $x_1 + x_2 + x_3 + \cdots + x_n = 1$
  \[
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
  \end{bmatrix}
  \]
- $V$ = the space of all real valued functions defined on the interval $[0, 1]$,
  $S$ = the space of all real valued discontinuous functions defined on the interval $[0, 1]$
- $V$ = the space of all real valued functions defined on the interval $[0, 1]$,
  $S$ = the space of those functions in $V$ that are monotonic increasing
- $V = C[-2, 2]$, $S$ = the space of all real valued continuous functions $f$ defined on the interval $[-2, 2]$ such that $f(-2) = f(2) = 100$
- $V = C[0, 1]$, $S$ = the space of all functions $f$ in $C[0, 1]$ satisfying \[ \int_0^1 f(x) \, dx = 3 \]
- $V = C[0, 1]$, $S$ = the set of all real polynomials of a single real variable with degree 2
- $V$ = the set of all sequences of real numbers,
  $S$ = the set of all sequences of real numbers \( \{a_n\} \) such that $\lim_{n \to \infty} a_n = 1$
EXERCISES:

With the usual definitions of matrix addition and a matrix multiplied by a scalar, are the following sets vector spaces?

1. The set of all real square matrices
2. The set of all real square matrices of size 2
3. The set of all real matrices of type $2 \times 3$
4. The set of all real square matrices of size 2 with nonegative entries
5. The set of all real square matrices of size 2 with 0 determinant
   (recall that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$)
6. The set of all real square matrices of size 2 with nonzero determinant
7. The set of all upper triangular real square matrices of size 3
   (recall that an upper triangular matrix is a matrix of the form $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$)

Turn over for the answers
Answers:

1. No  (Can we do a $2 \times 2$ matrix + a $3 \times 3$ matrix?)
2. Yes
3. Yes
4. No  (Examine $-2$ times $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$)
5. No  (Examine $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$)
6. No  (Is the 0 matrix in the set?)
7. Yes