Population Growth / Decay

Linear Exp. Model (Malthus, 1798)

Assumption Population y changes exponentially.

i.e. $\frac{dy}{dt} = ky$. $\quad y(t) = y_0 e^{kt}$.

Here, k = the net rate of change per unit of population

= (the birth rate per capita) − (the death rate per capita).

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Example A culture of bacteria grows exponentially.

Observed: # of bacteria increased 25 % in a hour.

Question: How long does it take for the # of bacteria to double?

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Example A culture of bacteria grows exponentially.

Observed: # of bacteria increased 25 % in a hour.

Question: How long does it take for the # of bacteria to double?

Solution: $y(t) = y_0 e^{kt}$.

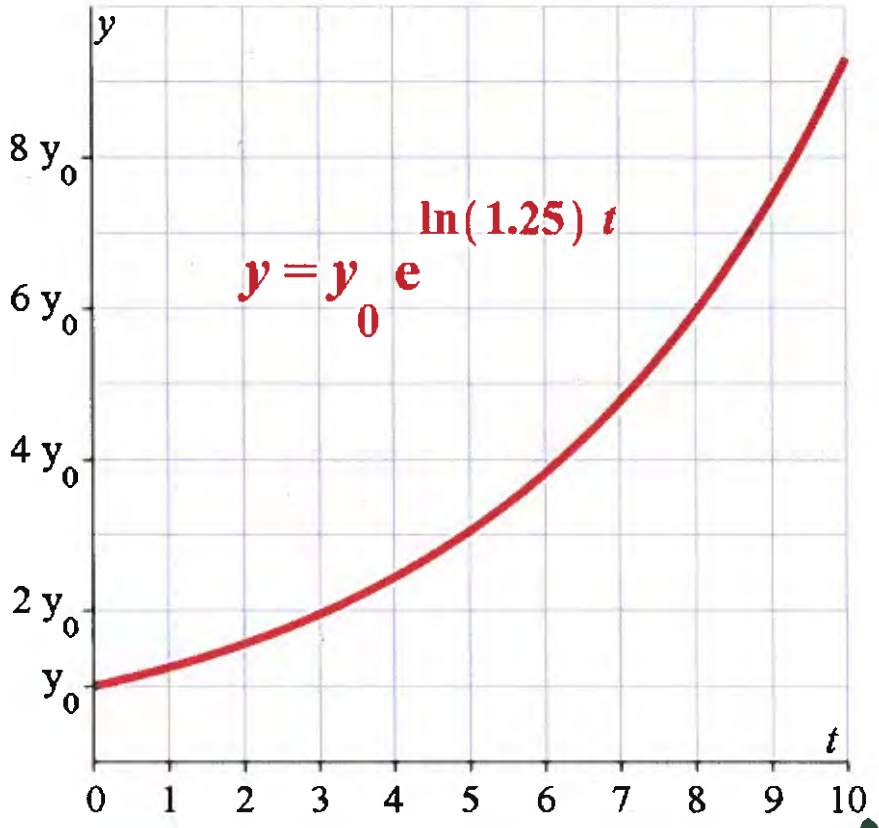
$y(1) = 125\, y_0 = y_0 e^{k \cdot 1} \Rightarrow 1.25 = e^k \Rightarrow k = \ln 1.25$

$y(t) = y_0 e^{(\ln 1.25)t}$.

Q. Find t such that $y(t) = 2y_0$.

$2y_0 = y_0 e^{(\ln 1.25)t} \Rightarrow 2 = e^{(\ln 1.25)t}$

$\Rightarrow \ln 2 = (\ln 1.25)t \Rightarrow t = \frac{\ln 2}{\ln 1.25}$ hours



Answer : Time to Double : $t_2 = \frac{\ln 2}{\ln 1.25} \approx 3.1$ hours

By the way: Time to Triple : $t_3 = \frac{\ln 3}{\ln 1.25} \approx 4.9$ hours

Time to Quadruple: $t_4 = \frac{\ln 4}{\ln 1.25} \approx 6.2$ hours

Time to Octuple : $t_8 = \frac{\ln 8}{\ln 1.25} \approx 9.3$ hours

we always have $t_4 = 2t_2$.

$t_8 = 3t_2$.

Q. Why?

Example

- Coffee Temp. $T(t)$, $T(0) = 190^\circ F$
- Room Temp. $70^\circ F$ (Surrounding Temp.)
  Constant

Newton's Law of Cooling

The rate of decrease of Coffee Temp.

$\propto$ Coffee Temp. − Room Temp.

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[An initial value problem of diff. eq.]

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- Room Temp. $70^\circ F$ (Surrounding Temp.) Constant

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The rate of decrease of Coffee Temp. $\propto$ Coffee Temp. − Room Temp.

$$-\frac{dT}{dt} = k(T-70), \quad T(0) = 190^\circ F.$$

[An initial value problem of diff. eq.]

Solution:

Set $y = T - 70$.

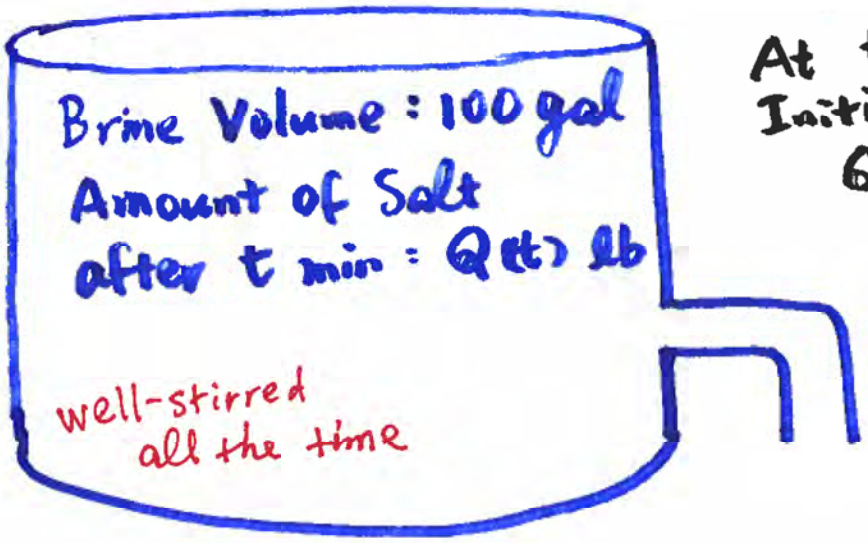
$$-\frac{dy}{dt} = ky. \qquad \begin{cases} \frac{dy}{dt} = -ky \\ y(0) = 190 - 70 = 120. \end{cases}$$

Solution: $y(t) = 120\,e^{-kt}$

Coffee Temp. $T(t) = 70 + y$

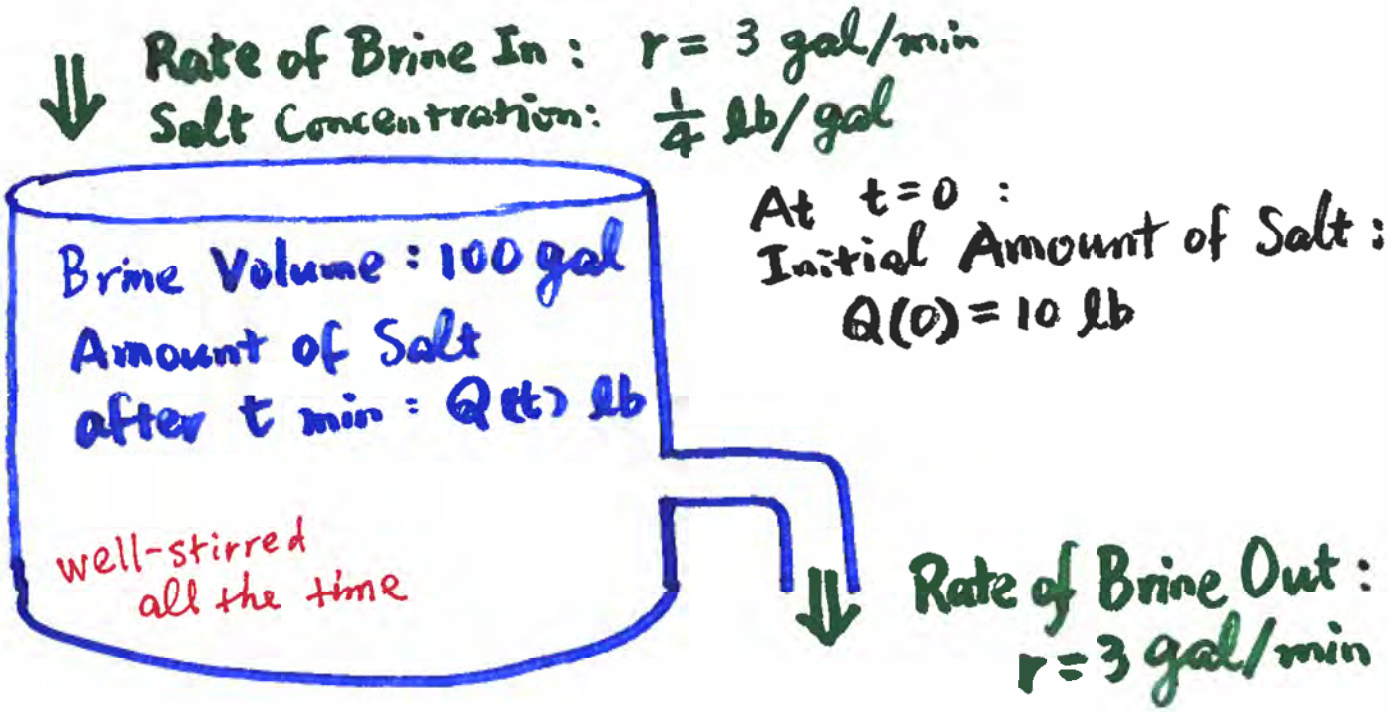
$= 70 + 120\,e^{-kt}$

Salt in a Tank



At $t = 0$:
Initial Amount of Salt :
$Q(0) = 10$ lb

Salt in a Tank



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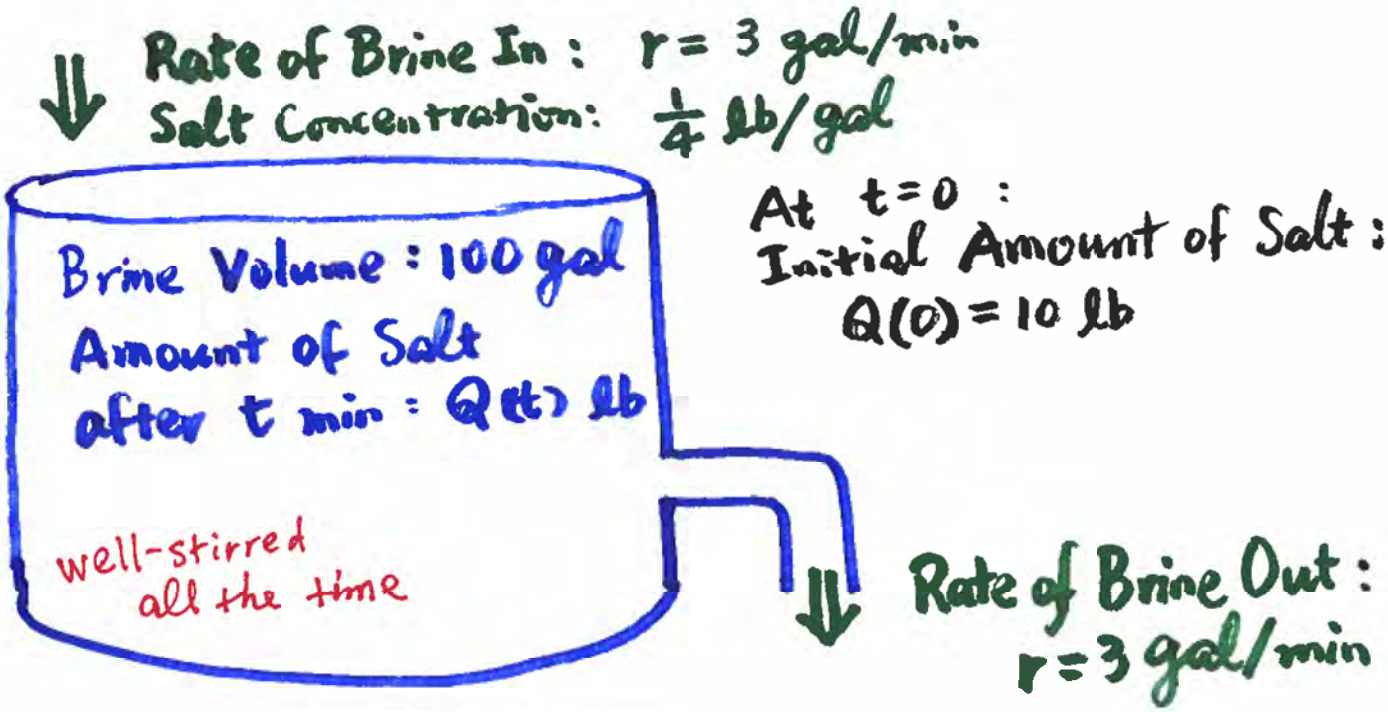
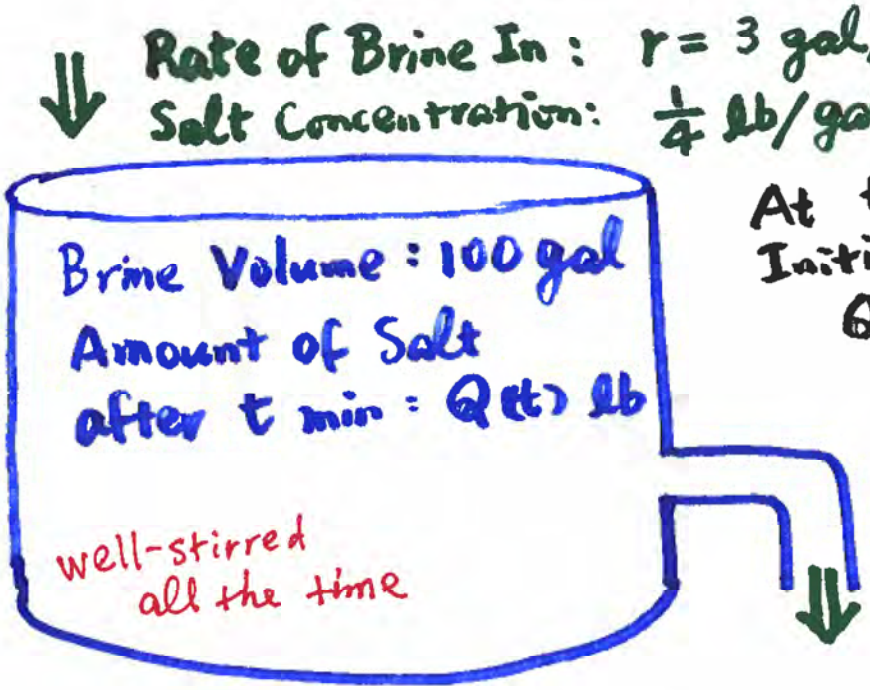


Question Find Q(t), the amount of Salt after t min.

Salt in a Tank



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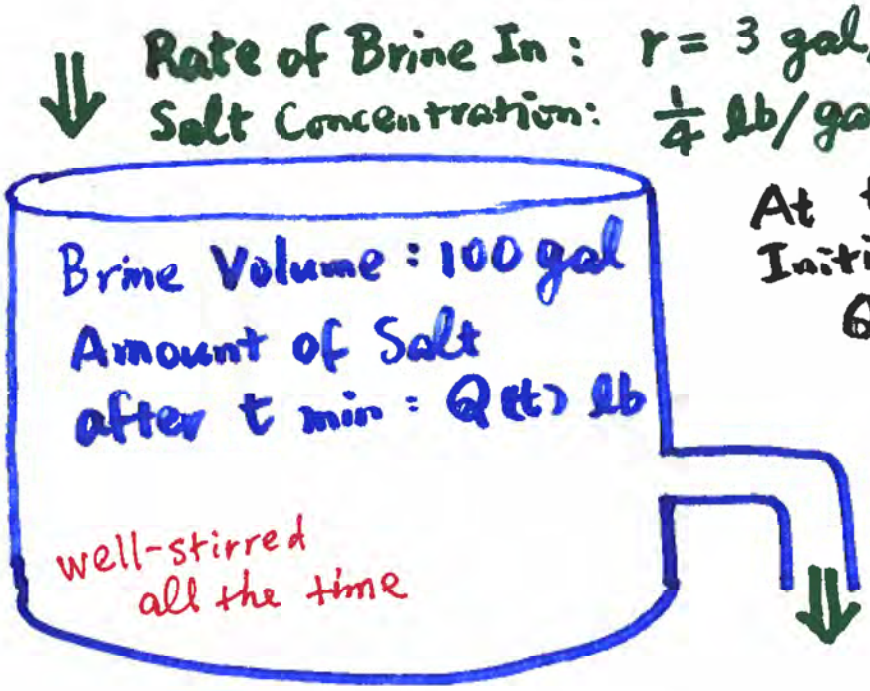
Set up a diff eq for $Q(t)$:

(rate of change of Salt) = (rate of Salt in) − (rate of Salt out)

$$\frac{dQ}{dt} = (3)\left(\frac{1}{4}\right) - (3)\left(\frac{Q}{100}\right)$$

Salt in a Tank



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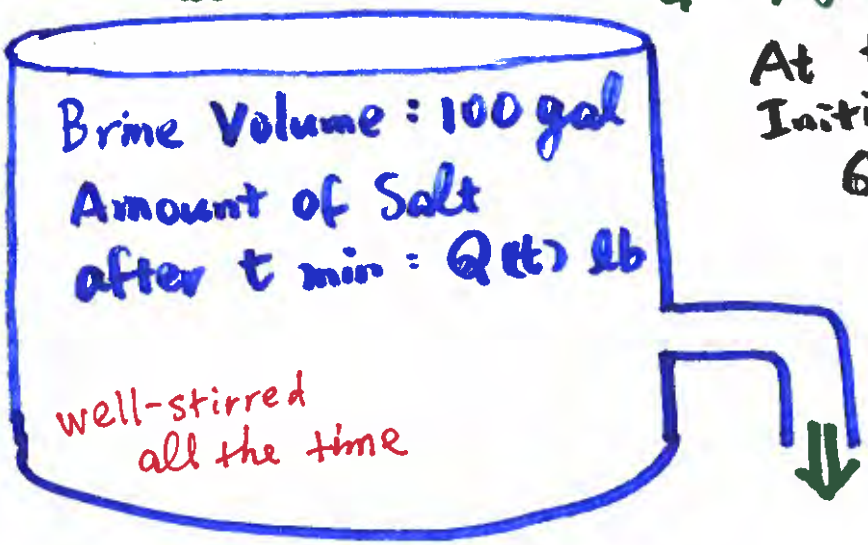
Diff Eq: $\frac{dQ}{dt} = \frac{3}{4} - \frac{3}{100}Q$ } Initial Value Problem

Init. Cond: $Q(0) = 10$

Salt in a Tank

Rate of Brine In: $r = 3$ gal/min
Salt Concentration: $\frac{1}{4}$ lb/gal



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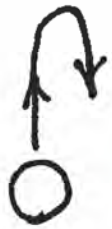
Solution Method 1: Separate the Variables Q and t.
Solution Method 2: Integrating Factor
Solution Method 3: Set $y = Q - 25$.

$$\begin{cases} \frac{dy}{dt} = -\frac{3}{100}y \\ y(0) = -15 \end{cases} \Rightarrow y = Ce^{-\frac{3}{100}t} = -15e^{-\frac{3}{100}t} \Rightarrow Q = 25 + y = 25 - 15e^{-\frac{3}{100}t} \text{ (lb)}.$$

A Ball Thrown Up & Falling Down



Newton's 2nd Law of Mechanics:

(Mass)(Acceleration) = (Net Force on the Mass)

- Assume Forces = { • Gravity mg ; • Air Resistance bv^2 }
- Let Upward Velocity > 0

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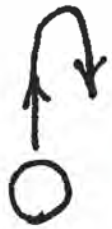

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Falling Down:

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$$\frac{dv}{dt} = -g - \frac{b}{m}v^2$$

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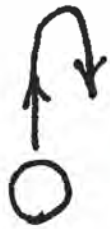

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Terminal Speed of Free Fall, v_T:

Objects	Terminal Speed
Basketball	20 m/s
Baseball	40 m/s
Ping Pong	9 m/s
Skydiver (parachute)	5-9 m/s
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Terminal Speed of Free Fall, v_T:

$$-g + \frac{b}{m}v_T^2 = 0 \Rightarrow v_T = \sqrt{\frac{mg}{b}}, \text{ or } \frac{b}{m} = \frac{g}{v_T^2}.$$

Up: $\frac{dv}{dt} = -g - \frac{g}{v_T^2}v^2$

Down: $\frac{dv}{dt} = -g + \frac{g}{v_T^2}v^2$

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Solve the diff eq:

Rising Up:

$$\frac{dv}{dt} = -g - \frac{g}{v_T^2} v^2$$

Separable Diff Eq

$$\int \frac{dv}{1+\frac{v^2}{v_T^2}} = \int -g\,dt, \quad v_T \arctan\frac{v}{v_T} = -gt + C,$$

$$v = v_T \tan\left(\frac{-gt+C}{v_T}\right)$$

Falling Down:

$$\frac{dv}{dt} = -g + \frac{g}{v_T^2} v^2$$

$$= \frac{g}{v_T^2}(v^2 - v_T^2)$$

$$\int \frac{dv}{v^2 - v_T^2} = \int \frac{g}{v_T^2}\,dt,$$

$$\frac{1}{v^2-v_T^2} = \frac{1}{2v_T}\left(\frac{1}{v-v_T} - \frac{1}{v+v_T}\right)$$

Partial Fractions

$$\frac{1}{2v_T}\left(\ln|v-v_T| - \ln|v+v_T|\right) = \frac{g}{v_T^2}t + C_1.$$

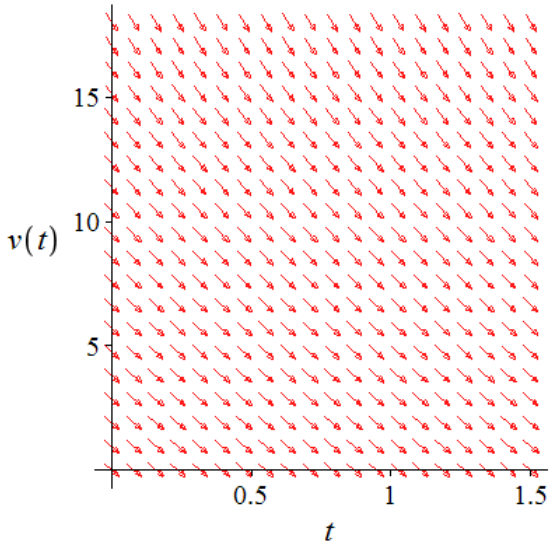
$$\frac{1}{2v_T}\ln\left|\frac{v-v_T}{v+v_T}\right|, \quad \ln\left|\frac{v-v_T}{v+v_T}\right| = \frac{2g}{v_T}t + 2v_T C_1$$

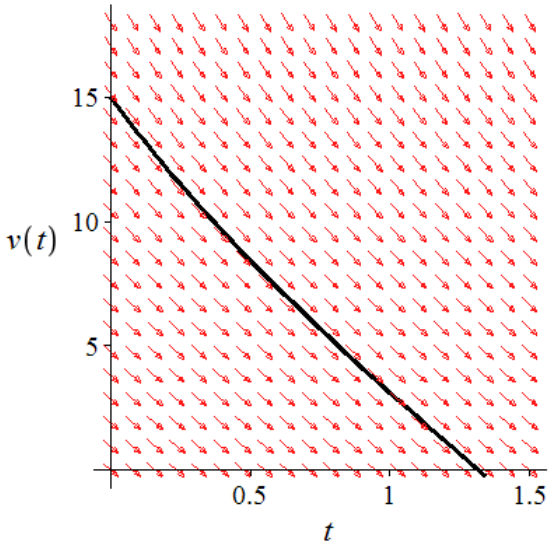
$$\frac{v-v_T}{v+v_T} = \pm e^{\frac{2g}{v_T}t + 2v_T C_1} = Ce^{\frac{2g}{v_T}t} \Rightarrow v = v_T \frac{1+Ce^{\frac{2g}{v_T}t}}{1-Ce^{\frac{2g}{v_T}t}}$$

Gravitational acceleration: $g = 9.8\ m/s^2$
Basketball terminal speed: $v_T = 20\ m/s$

Rising:

$$\frac{dv}{dt} = g\left(-1 - \frac{v^2}{v_T^2}\right)$$





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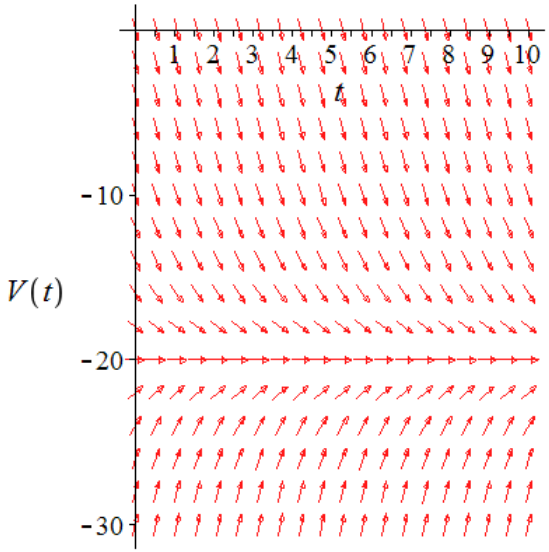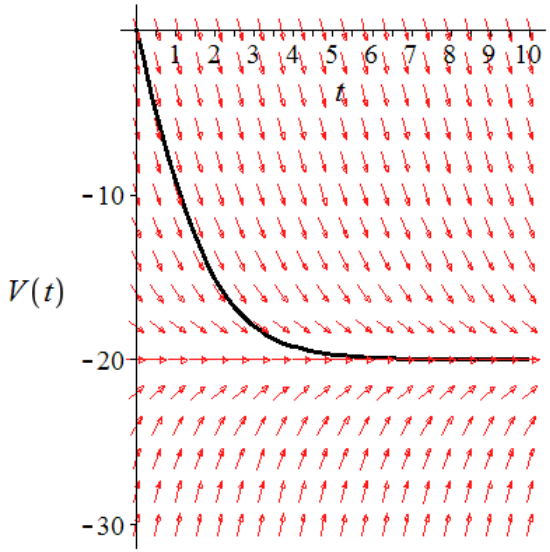
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