Linear Model of Population Dynamics

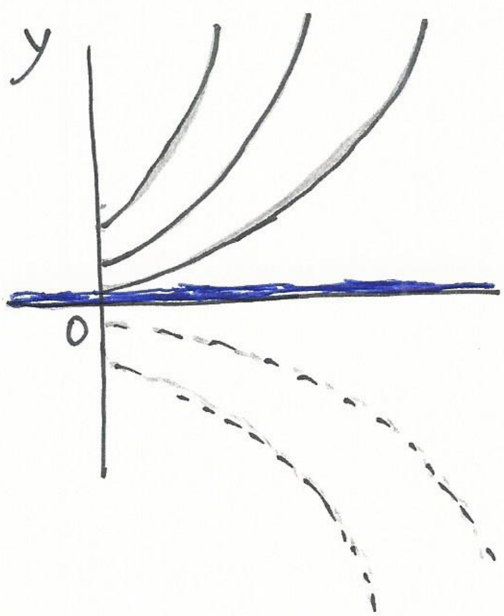
Malthus (1798)

Assumption

[The growth rate of a population $y(t)$]

= [a constant r]

$$\left.\begin{array}{ll} \underline{\text{Diff Eq}} & \dfrac{dy}{dt} = r\,y \\ \underline{\text{Init. Cond.}} & y(0) = y_0 \end{array}\right\} \Rightarrow \begin{array}{l}\text{Solution:} \\ y(t) = y_0 e^{rt}.\end{array}$$



When $r>0$: population grows without bound as $t\to\infty$.

- For $y(0)=0$, we have $y(t)\equiv 0$ for all t.
  (i.e. $y(t)=0$ is an equilibrium is a time-indep. sol.)
- For $y(0)\approx 0$, do we have $y(t)\approx 0$ for all $t>0$?

  Answer: No.

We say: $y(t)=0$ is an unstable equilibrium.

Nonlinear Logistic Model of Population Dynamics

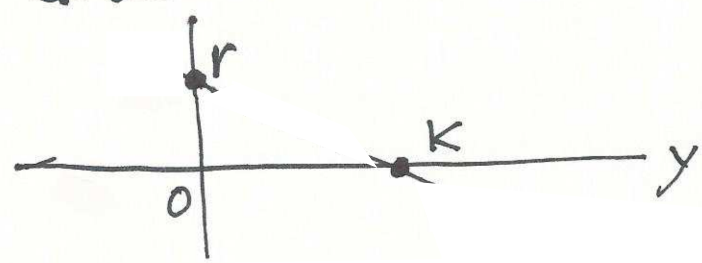
Verhulst (1838)

Assumptions

- Growth Rate depends on population y.
- $y \approx 0 \Rightarrow$ Growth Rate $\approx r \ (>0)$
- $y > K \Rightarrow$ Growth Rate < 0.

Simplest Choice:

Growth Rate vs. y



where $\begin{cases} r = \text{the intrinsic growth rate} & \text{(the growth rate when the resource limit is unimportant)} \\ K = \text{the carrying capacity} & \text{(the upper limit of the population that the resource can support)} \end{cases}$

Nonlinear Logistic Model of Population Dynamics

Verhulst (1838)

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$$\begin{cases} \cdot \text{ Growth Rate depends on population } y. \\ \cdot \; y \approx 0 \Rightarrow \text{Growth Rate} \approx r \; (>0) \\ \cdot \; y > K \Rightarrow \text{Growth Rate} < 0. \end{cases}$$

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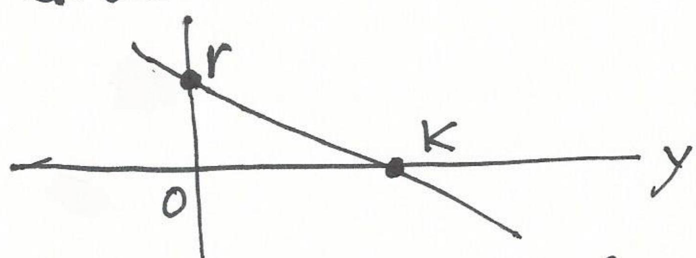

i.e. Growth Rate $= r\left(1 - \frac{y}{K}\right)$.

where $\begin{cases} r = \text{the intrinsic growth rate (the growth rate when the resource limit is unimportant)} \\ K = \text{the carrying capacity (the upper limit of the population that the resource can support)} \end{cases}$

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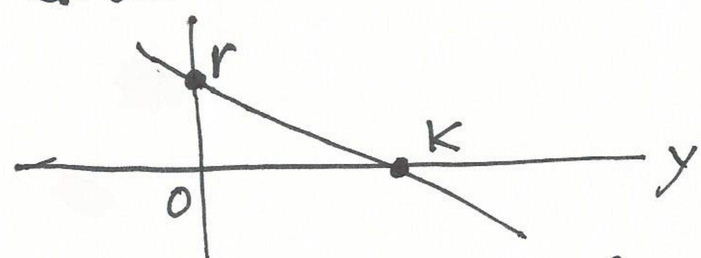

i.e. Growth Rate $= r\left(1-\frac{y}{K}\right)$.

$$\begin{cases} \text{Diff Eq.} & \dfrac{dy}{dt} = r\left(1-\dfrac{y}{K}\right)y \\ \text{Init. Cond.} & y(0) = y_0 \end{cases}$$

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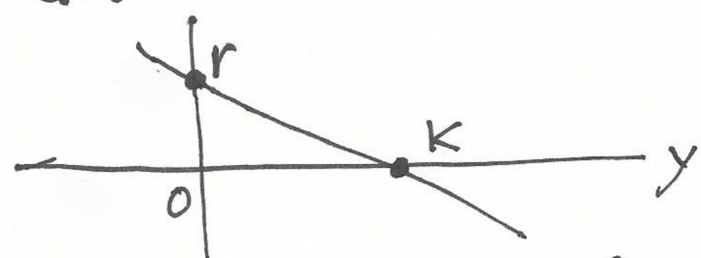

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$$\left\{\begin{array}{ll} \text{Diff Eq.} & \dfrac{dy}{dt} = r\left(1-\dfrac{y}{K}\right)y \\ \text{Init. Cond.} & y(0) = y_0 \end{array}\right\}$$

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Solution Formula

$$y(t) = \frac{K y_0}{y_0 + (K - y_0)e^{-rt}}$$

Example

$$\begin{cases} \dfrac{dy}{dt} = 0.03\left(1-\dfrac{y}{4}\right)y \\ y(0) = y_0 \end{cases}$$

Find the Solution $y(t)$.

Solution · The Diff Eq. is a separable eq.

· $$\frac{dy}{\left(1-\frac{y}{4}\right)y} = 0.03\,dt.$$

· Partial Fraction $$\frac{1}{\left(1-\frac{y}{4}\right)y} = \frac{4}{(4-y)y} = \frac{1}{4-y} + \frac{1}{y}$$

· $$\int \left[\frac{1}{4-y} + \frac{1}{y}\right] dy = \int 0.03\,dt.$$

$$-\ln|4-y| + \ln|y| = 0.03t + C,$$

$$\ln\left|\frac{y}{4-y}\right| = 0.03t + C, \quad \left|\frac{y}{4-y}\right| = e^{0.03t+C}$$

$$\frac{y}{4-y} = \pm e^{C} \cdot e^{0.03t}, \quad \frac{4-y}{y} = \pm e^{-C} \cdot e^{-0.03t}.$$

$$\frac{4}{y} - 1 = C_1 e^{-0.03t} \quad (\text{where } C_1 = \pm e^{-C})$$

Init. Cond. $y(0) = y_0 \Rightarrow \dfrac{4}{y_0} - 1 = C_1$

$$\frac{4}{y} - 1 = \left(\frac{4}{y_0} - 1\right)e^{-0.03t} = \frac{4-y_0}{y_0}\,e^{-0.03t}$$

$$\frac{4}{y} = 1 + \frac{4-y_0}{y_0}\,e^{-0.03t} = \frac{y_0 + (4-y_0)e^{-0.03t}}{y_0}$$

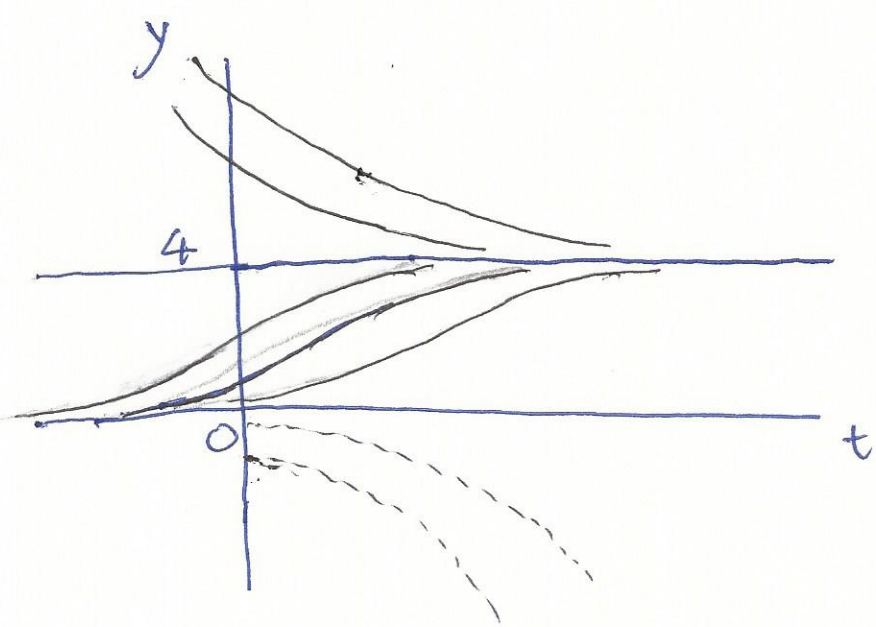
$$\boxed{y = \frac{4y_0}{y_0 + (4-y_0)e^{-0.03t}}}$$

Example $\dfrac{dy}{dt} = 0.03\left(1-\dfrac{y}{4}\right)y.$ $(r=0.03,\ K=4)$

Solution: $y(t) = \dfrac{4y_0}{y_0+(4-y_0)e^{-0.03t}}$

Solution Behavior:

- $y \equiv 0$ is an equilibrium (i.e. a Time-Indep. Sol.)
- $y \equiv 4$ is an equilibrium.
- If $y_0 > 4$, the sol. $y(t)$ decreases.
- If $0<y_0<4$, the sol. $y(t)$ increases.
- All positive solutions $y(t) \to 4$ as $t \to \infty$.



Stability, Asymptotic Stability, & Instability

- For $y(0)=0$, we have $y(t)\equiv 0$ for all t.
  i.e. $y=0$ is an <u>equilibrium</u> (is a time-indep. sol.)

Question:
For $y(0)\approx 0$, do we have $y(t)\approx 0$ for all $t>0$?

Answer NO.
We say: $y=0$ is an <u>unstable</u> equilibrium.

- For $y(0)=4$, we have $y(t)\equiv 4$ for all t.
  <u>i.e.</u> $y=4$ is an <u>equilibrium</u>

Question:
For $y(0)\approx 4$, do we have $y(t)\approx 4$ for all $t>0$?

Answer: Yes.
We say: $y=4$ is a <u>stable</u> equilibrium.

Question:
For $y(0)\approx 4$, do we have (i) $y(t)\approx 4$ for all $t>0$?
(ii) $\lim_{t\to\infty} y(t)=4$?

Answer: (i) Yes. (ii) Yes.
We say: $y=4$ is an <u>asymptotically stable</u> equilibrium

Stability, Asymptotic Stability, & Instability

$$\boxed{\frac{dy}{dt} = f(y)}$$

- [$y = b$ is an equilibrium] $\Leftrightarrow$ [$y = b$ is a time-indep. sol.]
  $\Leftrightarrow$ $0 = f(b)$
  $\Leftrightarrow$ [For $y(0) = b$, we have $y(t) = b$ for all t]

- Is $y = b$ an equilibrium? $\Leftrightarrow$ Is $f(b) = 0$?

- Is $y=b$ a <u>stable equilibrium</u>?
$\Leftrightarrow$ For $y(0) \approx b$, do we have $y(t) \approx b$ for all $t > 0$?

If Answer is yes, $y = b$ is a <u>stable</u> equilibrium.
If Answer is NO, $y = b$ is an <u>unstable</u> equilibrium.

- Is $y=b$ an <u>asymptotically stable equilibrium</u>?
$\Leftrightarrow$ For $y(0) \approx b$, do we have $\begin{cases} \text{(i) } y(t) \approx b \text{ for all } t > 0\text{?} \\ \text{(ii) } \lim_{t\to\infty} y(t) = b\text{?} \end{cases}$

If the answer (i) & the answer (ii)
are <u>both</u> yes,
$y = b$ is an <u>asymptotically stable</u> equilibrium