First Order Scalar Linear Diff Eq's

$$\frac{dy}{dt} + a(t)y = b(t)$$

unknown: $y(t)$

$a(t)$, $b(t)$ are given.

Examples

- $\dfrac{dy}{dt} + 4t^2 y = t^5$
- $\dfrac{dy}{dt} = y\sin(4t)$
- $\dfrac{dy}{dx} - 2y = 4 - x^2$ (unknown: $y(x)$)
- $t\dfrac{dy}{dt} + 2y = t^5$

Non-Examples

(First order nonlinear diff eq's)

- $\dfrac{dy}{dt} + 4ty^2 = t^5$
- $\dfrac{dy}{dt} = t\sin(4y)$
- $\dfrac{dy}{dx} - 2y = 4 - y^2$
- $t\dfrac{dy}{dt} + 2y = y^5$

(*) $\dfrac{dy}{dt} + a(t)y = b(t)$

Solution Method (integrating factor)

- Make sure the diff eq is a first order linear diff eq.
- Make sure (the coeff of $\frac{dy}{dt}$) equals 1. Otherwise divide to make it $=1$
- Prepare $A(t)$, an antiderivative of $a(t)$: $A'(t) = a(t)$.
  $A(t)$ is found by integrating $a(t)$.
- Eq (*) $\Leftrightarrow$ $e^{A(t)}\dfrac{dy}{dt} + e^{A(t)}a(t)y = b(t)e^{A(t)}$ $\Leftrightarrow$ $\dfrac{d}{dt}\left[e^{A(t)}y\right] = b(t)e^{A(t)}$
- $e^{A(t)}y = \int b(t)e^{A(t)}dt + C$
- $\boxed{y = e^{-A(t)}\int b(t)e^{A(t)}dt + Ce^{-A(t)}}$

$e^{A(t)}$ is called the integrating factor

(*) $\frac{dy}{dt} + a(t)y = b(t)$

Solution Method (integrating factor)

- Check: 1st order linear? (coeff of $\frac{dy}{dt}$) $= 1$?
- Prepare $A(t)$, an antiderivative of $a(t)$
- Eq (*) $\Leftrightarrow \frac{d}{dt}\left[e^{A(t)}y\right] = b(t)e^{A(t)}$
- $e^{A(t)}y = \int b(t)e^{A(t)}dt + C$
- $\boxed{y = e^{-A(t)}\int b(t)e^{A(t)}dt + Ce^{-A(t)}}$

Example 1 Solve $3\frac{dy}{dt} - 5y = (12-3t)e^{-\frac{1}{3}t}$

Solution $\Leftrightarrow \frac{dy}{dt} - \frac{5}{3}y = (4-t)e^{-\frac{1}{3}t}$

- $a(t) = -\frac{5}{3} \Rightarrow A(t) = -\frac{5}{3}t$
- Diff Eq $\Leftrightarrow \frac{d}{dt}\left(e^{-\frac{5}{3}t}y\right) = (4-t)e^{-2t}$
- $e^{-\frac{5}{3}t}y = \int (4-t)e^{-2t}dt = \int u\,dv = uv - \int v\,du$

Int. by Parts $\begin{cases} \text{Set } u = 4-t, & dv = e^{-2t}dt \\ \text{Then } du = -dt, & v = -\frac{1}{2}e^{-2t} \end{cases}$

$$= (4-t)\left(-\tfrac{1}{2}e^{-2t}\right) - \int\left(-\tfrac{1}{2}e^{-2t}\right)(-dt) = \left(-2+\tfrac{1}{2}t\right)e^{-2t} - \int \tfrac{1}{2}e^{-2t}dt$$

$$= \left(-2+\tfrac{1}{2}t\right)e^{-2t} + \tfrac{1}{4}e^{-2t} + C = \left(-\tfrac{7}{4}+\tfrac{1}{2}t\right)e^{-2t} + C$$

$$\boxed{y = \left(-\tfrac{7}{4}+\tfrac{1}{2}t\right)e^{-\frac{1}{3}t} + Ce^{\frac{5}{3}t}}$$ where C is any constant.

Example 1 Solve $3\frac{dy}{dt}-5y=(12-3t)\,e^{-\frac{1}{3}t}$ ← A Diff Eq.

Answer:

$$\boxed{y=\left(-\frac{7}{4}+\frac{1}{2}t\right)e^{-\frac{1}{3}t}+Ce^{\frac{5}{3}t}}$$

where C is any constant. (infinitely many solutions parametrized by C)

Example 2 Solve
$$\begin{cases} 3\frac{dy}{dt}-5y=(12-3t)\,e^{-\frac{1}{3}t} & \text{(Diff Eq)} \\ y(1)=e^{-\frac{1}{3}} & \text{(Initial Condition)} \end{cases}$$
An Initial Value Problem of a Diff Eq.

Solution

- The general sol's of the diff eq: $y=\left(-\frac{7}{4}+\frac{1}{2}t\right)e^{-\frac{1}{3}t}+Ce^{\frac{5}{3}t}$

- Initial Condition

$$y(1)=e^{-\frac{1}{3}} \Rightarrow e^{-\frac{1}{3}}=\left(-\frac{7}{4}+\frac{1}{2}\right)e^{-\frac{1}{3}}+Ce^{\frac{5}{3}} \Rightarrow \frac{9}{4}e^{-\frac{1}{3}}=Ce^{\frac{5}{3}}$$

$$\Rightarrow C=\frac{9}{4}e^{-2}$$

$$\Rightarrow \boxed{y=\left(-\frac{7}{4}+\frac{1}{2}t\right)e^{-\frac{1}{3}t}+\frac{9}{4}e^{-2}\,e^{\frac{5}{3}t}}$$

(the unique sol of the initial value problem)

(*) $\frac{dy}{dt} + a(t)y = b(t)$ <u>Solution Method</u> (integrating factor)

- Check: 1st order linear? (coeff of $\frac{dy}{dt}$) = 1 ?
- Prepare $A(t)$, an antiderivative of $a(t)$
- Eq (*) $\Leftrightarrow \frac{d}{dt}\left[e^{A(t)}y\right] = b(t)e^{A(t)}$
- $e^{A(t)}y = \int b(t)e^{A(t)}dt + C$
- $\boxed{y = e^{-A(t)}\int b(t)e^{A(t)}dt + Ce^{-A(t)}}$

Example 3 Solve $y' + 2xy = x^3$.

<u>Solution</u>

- $a(x) = 2x \Rightarrow A(x) = x^2$.
- The Diff Eq $\Leftrightarrow \frac{d}{dx}\left(e^{x^2}y\right) = x^3e^{x^2}$
- $e^{x^2}y = \int x^3e^{x^2}dx = \int x^2e^{x^2}\cdot x\,dx = \int se^s\cdot\frac{1}{2}ds = \int\frac{1}{2}se^s ds$

substitute $s = x^2$, $ds = 2x\,dx$

int. by Parts $\begin{cases} \text{Set } u = \frac{1}{2}s, & dv = e^s ds \\ \text{Then } du = \frac{1}{2}ds, & v = e^s \end{cases}$

$= \int u\,dv = uv - \int v\,du = \frac{1}{2}se^s - \int e^s\frac{1}{2}ds$

$= \frac{1}{2}se^s - \frac{1}{2}e^s + C = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$

$\boxed{y = \frac{1}{2}x^2 - \frac{1}{2} + Ce^{-x^2}}$

where C is any constant

Example 3 Solve $y' + 2xy = x^3$

Answer:

$$\boxed{y = \frac{1}{2}x^2 - \frac{1}{2} + Ce^{-x^2}}$$ where C is any constant

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Example 4 Solve $\begin{cases} y' + 2xy = x^3 \\ y(0) = 4 \end{cases}$

Solution

- The gen. sol's of the diff eq: $y = \frac{1}{2}x^2 - \frac{1}{2} + Ce^{-x^2}$
- Init. Cond. $\Rightarrow 4 = -\frac{1}{2} + C$

$\Rightarrow C = \frac{9}{2}$

$\Rightarrow \boxed{y = \frac{1}{2}x^2 - \frac{1}{2} + \frac{9}{2}e^{-x^2}}$ (The uniq. sol. of the initial value problem)

$$(*) \quad \frac{dy}{dt} + a(t)y = b(t)$$

<u>Solution Method</u> (integrating factor)

- Check: 1st order linear? (Coeff of $\frac{dy}{dt}$) = 1 ?
- Prepare $A(t)$, an antiderivative of $a(t)$
- Eq $(*) \iff \frac{d}{dt}\left[e^{A(t)}y\right] = b(t)e^{A(t)}$
- $e^{A(t)}y = \int b(t)e^{A(t)}dt + C$
- $\boxed{y = e^{-A(t)}\int b(t)e^{A(t)}dt + Ce^{-A(t)}}$

Example 5 Solve $t\frac{dy}{dt} = \pi t^2 e^{-t^2}\sin(\pi t) + (1-2t^2)y \quad (t>0)$.

<u>Solution</u> $\iff t\frac{dy}{dt} + (2t^2-1)y = \pi t^2 e^{-t^2}\sin(\pi t)$

$$\iff \frac{dy}{dt} + \left(2t - \frac{1}{t}\right)y = \pi t e^{-t^2}\sin(\pi t)$$

- $a(t) = 2t - \frac{1}{t} \Rightarrow A(t) = t^2 - \ln t \Rightarrow e^{A(t)} = e^{t^2-\ln t} = \frac{1}{t}e^{t^2}$
- The Diff Eq $\iff \frac{d}{dt}\left(\frac{1}{t}e^{t^2}y\right) = \frac{1}{t}e^{t^2}\pi t e^{-t^2}\sin(\pi t) = \pi\sin(\pi t)$
- $\frac{1}{t}e^{t^2}y = \int \pi\sin(\pi t)\,dt = -\cos(\pi t) + C$

$$\boxed{y = te^{-t^2}\left[-\cos(\pi t) + C\right]}$$

where C is any constant

Example 5 Solve $t\frac{dy}{dt} = \pi t^2 e^{-t^2}\sin(\pi t) + (1-2t^2)y \quad (t>0)$.

Answer $\boxed{y = te^{-t^2}[-\cos(\pi t) + C]}$

Example 6 Solve $\begin{cases} t\frac{dy}{dt} = \pi t^2 e^{-t^2}\sin(\pi t) + (1-2t^2)y \quad (t>0) \\ y(1) = 2. \end{cases}$

Solution

- The gen. sol's of the diff eq: $y = te^{-t^2}[-\cos(\pi t) + C]$
- Init. Cond. $\Rightarrow$ $2 = e^{-1}[-\cos\pi + C]$

$\cos\pi = -1$

$\Rightarrow C = 2e - 1$

$\Rightarrow \boxed{y = te^{-t^2}[-\cos(\pi t) + 2e - 1]}$ (The uniq. sol. of the initial value problem)