2nd order diff equations

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Examples

① $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 15y = 0$ — Homog. Lin. Const. Coeff.

② $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 15y = 4e^{-t}\sin(2t)$ — Nonhomog. Lin. Const. Coeff.

③ $y'' + (\cos t - 2)y = 0$ — Homog Lin. Variable Coeff.

④ $y'' + (\cos t - 2)y = \sin(5t)$ — Nonhomog. Lin. Variable Coeff.

⑤ $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + \sin y = 0$ — Nonlinear

⑥ $y'' + y^3 = \sin(2t)$ — Nonlinear

⑦ $y'' + 3y' + \sin(t+y) = 0$ — Nonlinear

Convert a 2nd Order Diff Eq to a 2-dim. System of Diff Eq's.

Example. Given (*) $3y'' + 5ty' - 6y = 0$ (a 2nd ord. homog. lin. diff eq.)

Set $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$

Diff Eq for x_1: $x_1' = y'$ (by the definition of x_1)

$= x_2$ (by the definition of x_2)

Diff Eq for x_2: $x_2' = y''$ (by the def. of x_2)

$= 2y - \frac{5}{3}ty'$ (by the given diff eq (*))

$= 2x_1 - \frac{5}{3}tx_2$ (by the def. of x_1, x_2).

2-D System. (Homog. Lin).

$$\begin{cases} x_1' = x_2 \\ x_2' = 2x_1 - \frac{5}{3}tx_2 \end{cases}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 2 & -\frac{5}{3}t \end{bmatrix} \vec{x}$$

2nd ord Homog. Lin Diff Eq $\quad a_2(t)\frac{d^2y}{dt^2}+a_1(t)\frac{dy}{dt}+a_0(t)y=0 \quad (*)$

$\Updownarrow$ Set $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$

2-D Homog. Lin System of Diff Eqs $\quad \frac{d\vec{x}}{dt}=A(t)\vec{x} \quad (**)$

$$A(t)=\begin{bmatrix} 0 & 1 \\ -\frac{a_0(t)}{a_2(t)} & -\frac{a_1(t)}{a_2(t)} \end{bmatrix}$$

- If $y(t)$ is a sol of $(*)$, then $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$ is a sol. of $(**)$.

- If $\vec{x}(t)$ is a sol of $(**)$, then [the first component of $\vec{x}(t)$] gives a sol. $y(t)$ of $(*)$.

- Gen. Sol's of $(**)$ are: $\vec{x}(t)=c_1\left[\overrightarrow{\text{Sol.1}}\right]+c_2\left[\overrightarrow{\text{Sol.2}}\right]$.

Thus,
Gen. Sol's of $(*)$ are: $y(t)=c_1[\quad]+c_2[\quad]$

The first component of $\overrightarrow{\text{Sol.1}}$

The first component of $\overrightarrow{\text{Sol.2}}$

2nd order Homogeneous Linear Diff Eq's

$$a_2(t)\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = 0$$

Gen. Solutions $\quad y(t) = C_1\, y_1(t) + C_2\, y_2(t)$

Two Linearly Independent Solutions.

Solution Method does not exist in general, when $a_2(t)$, $a_1(t)$, $a_0(t)$ are t dependent.

2nd order Homog. Lin. Diff Eq's with Constant Coefficients

$$a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = 0$$

where $a_2 \neq 0$.
a_2, a_1, a_0 are constants

can be solved by eigenvalues.

If λ is an eigenvalue, then $e^{\lambda t}$ is a solution of the 2nd order homog. lin. diff. eq. with Constant Coeff.

Example A 2nd order homog. lin. diff eq. with Constant Coefficients

$$3y'' + 14y' + 8y = 0.$$

Set $\begin{cases} x_1 = y \\ x_2 = y' \end{cases} \Rightarrow \begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = -\frac{8}{3}y - \frac{14}{3}y' = -\frac{8}{3}x_1 - \frac{14}{3}x_2 \end{cases}$

$\Rightarrow$ A 2-D Homog. Lin. System with Constant Coefficients $\quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{8}{3} & -\frac{14}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$

- Eigenvalues $\quad \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -\frac{8}{3} & -\frac{14}{3} - \lambda \end{bmatrix} = \lambda^2 + \frac{14}{3}\lambda + \frac{8}{3} = 0.$

$$3\lambda^2 + 14\lambda + 8 = 0 \xrightarrow[\text{solve}]{} \lambda_1 = -\frac{2}{3}, \quad \lambda_2 = -4.$$

- Eigenvectors for $\lambda_1 = -\frac{2}{3}$

$(A - \lambda_1 I)\vec{x} = \vec{0}, \quad (A + \frac{2}{3}I)\vec{x} = \vec{0}, \quad \begin{bmatrix} \frac{2}{3} & 1 \\ -\frac{8}{3} & -\frac{12}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \frac{2}{3}x_1 + x_2 = 0 \Leftrightarrow x_1 = -\frac{3}{2}x_2.$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} \Rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$ is an eigenvector for $\lambda_1 = -\frac{2}{3}$.

$\Rightarrow$ A particular sol of the 2-D system: $\vec{x}(t) = e^{\lambda_1 t}\vec{u}_1, \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-\frac{2}{3}t}\begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}.$

$\Rightarrow$ A particular sol. of the 2nd ord. diff eq: $y(t) = x_1(t) = e^{-\frac{2}{3}t}.$

In general,

[2nd order homog. lin. diff. eq. with Const. Coeff

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0$$
] can be converted to [2D Homog. Lin. System with Const. Coeff.

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0&1\\ * & * \end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$
]

Set $\begin{cases}x_1=y\\x_2=y'\end{cases}$

- Eigenvalues : $a\lambda^2+b\lambda+c=0$
- Eigenvectors : If λ is an eigenvalue, then $\begin{bmatrix}1\\\lambda\end{bmatrix}$ is a corresponding eigenvector.
- Exp. Solutions : If λ is an eigenvalue, then $e^{\lambda t}$ is a solution of the y equation.
- Gen. Solutions :
  - $\lambda_1\neq\lambda_2 \Rightarrow y(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$
  - $\lambda_1=\lambda_2 \Rightarrow y(t)=C_1e^{\lambda_1 t}+C_2te^{\lambda_1 t}$
  - $\lambda_{1,2}=\alpha\pm i\beta \Rightarrow y(t)=C_1e^{\alpha t}\cos(\beta t)+C_2e^{\alpha t}\sin(\beta t)$