Second Order Nonhomogeneous Linear Differential Equations with Constant Coefficients: 
the method of undetermined coefficients

Xu-Yan Chen
Second Order Nonhomogeneous Linear Differential Equations with Constant Coefficients:

\[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t), \]

where \( a_2 \neq 0, a_1, a_0 \) are constants, and \( f(t) \) is a given function (called the nonhomogeneous term).

General solution structure:

\[ y(t) = y_p(t) + y_c(t) \]

where \( y_p(t) \) is a particular solution of the nonhomog equation, and \( y_c(t) \) are solutions of the homogeneous equation:

\[ a_2 y_c''(t) + a_1 y_c'(t) + a_0 y_c(t) = 0. \]

The characteristic roots: \( a_2 \lambda^2 + a_1 \lambda + a_0 = 0 \)

\( \Rightarrow \) The complementary solutions \( y_c(t) \).
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What is this note about?  The Method of Undetermined Coefficients:
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What is this note about? **The Method of Undetermined Coefficients:** a method of finding \( y_p(t) \), when the nonhomog term \( f(t) \) belongs a simple class.
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**What is this note about?** The Method of Undetermined Coefficients: a method of finding \( y_p(t) \), when the nonhomog term \( f(t) \) belongs a simple class.

**Main Idea:** Set up a trial function \( y_p(t) \), by copying the function form of \( f(t) \).
Example 1: Solve $3y'' + y' - 2y = 10e^{4t}$, \( y(0) = -1, y'(0) = 3 \).
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- Find complementary solutions \( y_c(t) \):
  
  \[
  3y''_c + y'_c - 2y_c = 0 \quad \text{(the corresponding homog eq)}
  \]
  
  \[
  3\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = \frac{2}{3} \Rightarrow y_c = C_1 e^{-t} + C_2 e^{\frac{2}{3}t}
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  $3y''_c + y'_c - 2y_c = 0$ (the corresponding homog eq)

  $3\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 2/3 \Rightarrow y_c = C_1e^{-t} + C_2e^{2/3t}$

- To find $y_p(t)$, set the trial function

  $y_p(t) = ae^{4t}$ (form copied from $f(t) = 10e^{4t}$)

where $a$ is the undetermined coefficient.
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- Substitute $y_p(t)$ in the nonhomog eq:

  
  \[
  3(ae^{4t})'' + (ae^{4t})' - 2ae^{4t} = 10e^{4t}
  \]

  \[
  = 3(16ae^{4t}) + (4ae^{4t}) - 2ae^{4t}
  \]

  \[
  = 50ae^{4t}
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  \[= 50ae^{4t}\]
- Compare the coefficients of the two sides:
  \[50a = 10 \Rightarrow a = \frac{1}{5}\]
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  $y_p(t) = ae^{4t}$ \quad (form copied from $f(t) = 10e^{4t}$)

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  $= 50ae^{4t}$

- Compare the coefficients of the two sides:

  $50a = 10 \Rightarrow a = \frac{1}{5} \Rightarrow y_p(t) = \frac{1}{5}e^{4t}$
Example 1 (continued): Solve
\[3y'' + y' - 2y = 10e^{4t}, \quad y(0) = -1, y'(0) = 3.\]

Combine \(y_c\) and \(y_p\) to get

Gen Sols of Nonhomg Eq: \[y(t) = \frac{1}{5}e^{4t} + C_1e^{-t} + C_2e^{\frac{2}{3}t}.\]
Example 1 (continued): Solve
\[ 3y'' + y' - 2y = 10e^{4t}, \quad y(0) = -1, y'(0) = 3. \]

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\text{Gen Sols of Nonhomg Eq: } \quad y(t) = \frac{1}{5}e^{4t} + C_1 e^{-t} + C_2 e^{\frac{2}{3}t}.
\]

▷ Use initial conditions:

\[
y(0) = -1 \quad \Rightarrow \quad \frac{1}{5} + C_1 + C_2 = -1
\]
\[
y'(t) = \frac{4}{5}e^{4t} - C_1 e^{-t} + \frac{2}{3}C_2 e^{\frac{2}{3}t}, \quad y'(0) = 3 \quad \Rightarrow \quad \frac{4}{5} - C_1 + \frac{2}{3}C_2 = 3
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Example 1 (continued): Solve
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Solve this:

\[
\begin{cases}
C_1 = -\frac{9}{5} \\
C_2 = \frac{3}{5}
\end{cases}
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Gen Sols of Nonhomg Eq:
\[ y(t) = \frac{1}{5}e^{4t} + C_1 e^{-t} + C_2 e^{\frac{2}{3}t}. \]

- Use initial conditions:

\[ y(0) = -1 \quad \Rightarrow \quad \frac{1}{5} + C_1 + C_2 = -1 \]
\[ y'(t) = \frac{4}{5}e^{4t} - C_1 e^{-t} + \frac{2}{3}C_2 e^{\frac{2}{3}t}, \quad y'(0) = 3 \quad \Rightarrow \quad \frac{4}{5} - C_1 + \frac{2}{3}C_2 = 3 \]

Solve this:
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- The solution of the initial value problem:

\[ y(t) = \frac{1}{5}e^{4t} - \frac{9}{5} e^{-t} + \frac{3}{5} e^{\frac{2}{3}t}. \]
Nonhomogeneous Linear Equations:

\[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t), \]

Towards the Rules of Setting Up the Trial Function:

\[
\begin{array}{c|c}
  f(t) & y_p(t) \\
  \hline
  k e^{rt} & A e^{rt} \\
  \hline
  \ldots & \ldots \\
\end{array}
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(to be continued)
Example 2: Solve \(3y'' + y' - 2y = -8te^{-2t}\).
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Complementary solutions \(y_c(t)\):

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3\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 2/3 \Rightarrow y_c = C_1 e^{-t} + C_2 e^{2/3t}
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- To find $y_p(t)$, set the trial function
  
  \[y_p(t) = At e^{-2t}.\]
Example 2: Solve $3y'' + y' - 2y = -8te^{-2t}$.

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- To find $y_p(t)$, set the trial function
  \[y_p(t) = Ae^{-2t}.\]

- Substitute $y_p(t)$ in the nonhomog eq:
  \[-8te^{-2t} = 3(Ate^{-2t})'' + (Ate^{-2t})' - 2Ate^{-2t}\]
  \[= 3(-4Ae^{-2t} + 4Ate^{-2t}) + (Ae^{-2t} - 2Ate^{-2t}) - 2Ate^{-2t}\]
  \[= -11Ae^{-2t} + 8Ate^{-2t}\]
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- Compare the coefficients of the two sides:

  $$\begin{cases} 
  -11A = 0 \\
  8A = -8
  \end{cases} \Rightarrow \text{Impossible!}$$
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  $$y_p(t) = Ate^{-2t}.$$  \textbf{Wrong!}

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- Compare the coefficients of the two sides:

  $$\left\{ \begin{array}{l}
  -11A = 0 \\
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  \end{array} \right. \Rightarrow \text{Impossible!}$$

  The choice of the trial function $y_p(t) = Ate^{-2t}$ was \textbf{WRONG}!
Example 2 (continued): Solve $3y'' + y' - 2y = -8te^{-2t}$.

The correct point of view:

$$f(t) = -8te^{-2t} = (a \text{ polynomial of degree one})e^{-2t}.$$
Example 2 (continued): Solve $3y'' + y' - 2y = -8te^{-2t}$.

- The correct point of view:
  
  \[ f(t) = -8te^{-2t} = (\text{a polynomial of degree one})e^{-2t}. \]

- The correct trial function:
  
  \[ y_p(t) = (A + Bt)e^{-2t}. \]
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- **Substitute** $y_p(t)$ **in the nonhomog eq:**
  \[
  -8te^{-2t} = 3[(A + Bt)e^{-2t}]'' + [(A + Bt)e^{-2t}]' - 2(A + Bt)e^{-2t}
  = 3(4A - 4B + 4Bt)e^{-2t} + (-2A + B - 2Bt)e^{-2t}
  + (-2A - 2Bt)e^{-2t}
  = (8A - 11B)e^{-2t} + 8Bte^{-2t}
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- Substitute $y_p(t)$ in the nonhomog eq:
  
  
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  &\quad + (-2A - 2Bt)e^{-2t} \\
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  \end{align*}

- Compare the coefficients of the two sides:

\[
\begin{cases}
  8A - 11B = 0 \\
  8B = -8
\end{cases} \quad \Rightarrow \quad \begin{cases}
  A = -\frac{11}{8} \\
  B = -1
\end{cases} \quad \Rightarrow \quad y_p(t) = \left(-\frac{11}{8} - t\right)e^{-2t}
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  \end{cases} \quad \Rightarrow \quad \begin{cases}
  A = -\frac{11}{8} \\
  B = -1
  \end{cases} \quad \Rightarrow \quad y_p(t) = \left( -\frac{11}{8} - t \right) e^{-2t}
  \]

- The General Solutions of the Nonhomogeneous Equation:
  
  \[ y(t) = y_p(t) + y_c(t) = \left( -\frac{11}{8} - t \right) e^{-2t} + C_1e^{-t} + C_2e^{\frac{3}{2}t}. \]
Example 3: Solve $3y'' + y' - 2y = -12t^2$. 
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- Complementary solutions $y_c(t)$:
  
  $3\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 2/3 \Rightarrow y_c = C_1 e^{-t} + C_2 e^{2/3 t}$
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- To find $y_p(t)$, set the trial function
  \[y_p(t) = At^2.\]
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  $y_p(t) = At^2$. This does not work!
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- To find $y_p(t)$, set the trial function
  \[y_p(t) = At^2. \quad \text{This does not work!}\]

- The correct trial function:
  \[y_p(t) = A + Bt + Ct^2.\]
Example 3: Solve $3y'' + y' - 2y = -12t^2$.

- Complementary solutions $y_c(t)$:
  
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- Substitute $y_p(t)$ in the nonhomog eq:
  
  $$-12t^2 = 3(2C) + (B + 2Ct) - 2(A + Bt + Ct^2)$$
  
  $$= (-2A + B + 6C) + (-2B + 2C)t - 2Ct^2$$
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- Substitute $y_p(t)$ in the nonhomog eq:
  
  $-12t^2 = 3(2C) + (B + 2Ct) - 2(A + Bt + Ct^2)$

  $= (-2A + B + 6C) + (-2B + 2C)t - 2Ct^2$

- Compare the coefficients of the two sides:
  
  $\begin{cases} -2A + B + 6C = 0 \\ -2B + 2C = 0 \Rightarrow B = 6 \Rightarrow y_p(t) = 21 + 6t + 6t^2 \end{cases}$
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- Complementary solutions $y_c(t)$:
  \[ 3\lambda^2 + \lambda - 2 = 0 \implies \lambda_1 = -1, \lambda_2 = 2/3 \implies y_c = C_1 e^{-t} + C_2 e^{2/3 t} \]

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  = (-2A + B + 6C) + (-2B + 2C)t - 2Ct^2
  \]

- Compare the coefficients of the two sides:
  \[
  \begin{cases}
  -2A + B + 6C = 0 & A = 21 \\
  -2B + 2C = 0 & B = 6 \implies y_p(t) = 21 + 6t + 6t^2 \\
  -2C = -12 & C = 6
  \end{cases}
  \]

- The General Solutions of the Nonhomogeneous Equation:
  \[ y(t) = y_p(t) + y_c(t) = 21 + 6t + 6t^2 + C_1 e^{-t} + C_2 e^{2/3 t}. \]
Nonhomogeneous Linear Equations:

\[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t), \]

Towards the Rules of Setting Up the Trial Function:

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( y_p(t) )</th>
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  \[
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  = -14A \cos(2t) - 2A \sin(2t)
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  The choice of the trial function $y_p(t) = A \cos(2t)$ was **Wrong**!
Example 4 (continued): Find a particular solution of 
$3y'' + y' - 2y = 5 \cos(2t)$.

The correct trial function:

$$y_p(t) = A \cos(2t) + B \sin(2t).$$
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Substitute $y_p(t)$ in the nonhomog eq:
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5 \cos(2t) = 3[A \cos(2t) + B \sin(2t)]'' + [A \cos(2t) + B \sin(2t)]'
- 2[A \cos(2t) + B \sin(2t)]
\]
\[
= 3[-4A \cos(2t) - 4B \sin(2t)] + [-2A \sin(2t) + 2B \cos(2t)]
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  \begin{cases}
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- **A Particular Solution of the Nonhomogeneous Equation:**
  \[y_p(t) = -\frac{7}{20} \cos(2t) + \frac{1}{20} \sin(2t).\]
Nonhomogeneous Linear Equations:

\[ a_2y''(t) + a_1y'(t) + a_0y(t) = f(t) \]

Towards the Rules of Setting Up the Trial Function:

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Example 5: Find a particular solution of

\[3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5 \cos(2t) + 17e^{-t} \cos t + 34e^{-t} \sin t\]
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We give two methods.
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Method 1:
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Solve $3y''_4 + y'_4 - 2y_4 = 5\cos(2t)$ to get a particular solution $y_4(t)$.
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A particular solution to the original equation:

\[
y_p(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t) = \frac{1}{5}e^{2t} + \left( -\frac{11}{8} - t \right) e^{-2t} + 21 + 6t + 6t^2 - \frac{7}{20} \cos(2t) + \frac{1}{20} \sin(2t) + \frac{7}{2}e^{-t} \cos t - \frac{11}{2}e^{-t} \sin t.
\]
Example 5 (continued): Find a particular solution of
\[ 3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5\cos(2t) + 17e^{-t}\cos t + 34e^{-t}\sin t \]

Method 2:

- Set a **BIIIIIG** trial function:
  \[ y_p(t) = A_0 e^{4t} + (A_1 + A_2t)e^{-2t} + (A_3 + A_4t + A_5t^2) \]
  \[ + [A_6 \cos(2t) + A_7 \sin(2t)] + (A_8 e^{-t}\cos t + A_9 e^{-t}\sin t), \]
  with undetermined coefficients \( A_0, A_1, \cdots, A_9 \).
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\[3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5 \cos(2t) + 17e^{-t} \cos t + 34e^{-t} \sin t\]

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- Set a BIG trial function:
  \[y_p(t) = A_0 e^{4t} + (A_1 + A_2 t)e^{-2t} + (A_3 + A_4 t + A_5 t^2) + [A_6 \cos(2t) + A_7 \sin(2t)] + (A_8 e^{-t} \cos t + A_9 e^{-t} \sin t),\]
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- Substitute this in the original equation.

- Compare the coefficients of the two sides
  \[\Rightarrow\] Linear equations for \(A_0, A_1, \cdots A_9\).
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- Substitute this in the original equation.

- Compare the coefficients of the two sides
  \[ \Rightarrow \text{Linear equations for } A_0, A_1, \cdots A_9. \]

- Solve \( A_0, A_1, \cdots A_9 \).
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- Substitute this in the original equation.

- Compare the coefficients of the two sides
  \[\Rightarrow\] Linear equations for \(A_0, A_1, \ldots A_9\).

- Solve \(A_0, A_1, \ldots A_9\).

- Finally obtain the particular solution

  \[y_p(t) = \frac{1}{5}e^{2t} + \left(-\frac{11}{8} - t\right)e^{-2t} + 21 + 6t + 6t^2\]
  \[-\frac{7}{20}\cos(2t) + \frac{1}{20}\sin(2t) + \frac{7}{2}e^{-t}\cos t - \frac{11}{2}e^{-t}\sin t.\]

(Computational details skipped here.)
Example 6: Find general solutions of $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$. 
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- To find \( y_p(t) \), set the trial function
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  36e^{3t} + 2e^{2t} = (Ae^{3t} + Be^{2t})'' - (Ae^{3t} + Be^{2t})' - 2(Ae^{3t} + Be^{2t}) \\
  = (9Ae^{3t} + 4Be^{2t}) - (3Ae^{3t} + 2Be^{2t}) - 2(Ae^{3t} + Be^{2t}) \\
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  \]
  \[
  = 4Ae^{3t}
  \]

- Compare the coefficients of the two sides:
  \[
  \begin{cases}
  4A = 36 \\
  0 = 2
  \end{cases} \implies \text{Impossible!}
  \]
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- Complementary solutions \( y_c(t) \):
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- To find \( y_p(t) \), set the trial function
  \[
  y_p(t) = Ae^{3t} + Be^{2t}.
  \]

  Wrong!

- Substitute \( y_p(t) \) in the nonhomog eq:
  \[
  36e^{3t} + 2e^{2t} = \left( Ae^{3t} + Be^{2t} \right)'' - \left( Ae^{3t} + Be^{2t} \right)' - 2\left( Ae^{3t} + Be^{2t} \right) = (9Ae^{3t} + 4Be^{2t}) - (3Ae^{3t} + 2Be^{2t}) - 2\left( Ae^{3t} + Be^{2t} \right) = 4Ae^{3t}
  \]

- Compare the coefficients of the two sides:
  \[
  \begin{cases}
  4A = 36 \\
  0 = 2
  \end{cases} \Rightarrow \text{Impossible!}
  \]

The trial function \( y_p(t) = Ae^{3t} + Be^{2t} \) was BAD!
Example 6 (continued): \[ y'' - y' - 2y = 36e^{3t} + 2e^{2t}. \]

- Complementary solutions: \( y_c = C_1 e^{2t} + C_2 e^{-t} \)
- The **BAD** trial function: \( y_p(t) = Ae^{3t} + Be^{2t}. \)
Example 6 (continued): \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions: \( y_c = C_1e^{2t} + C_2e^{-t} \)
- The BAD trial function: \( y_p(t) = Ae^{3t} + Be^{2t} \).
- The Reason of the Failure:
Example 6 (continued): \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

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  - When \( y_p(t) \) is plugged in the nonhomog eq, we wish the left hand side would match the right hand side \( 36e^{3t} + 2e^{2t} \).
Example 6 (continued): $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

- Complementary solutions: $y_c = C_1e^{2t} + C_2e^{-t}$
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- The Reason of the Failure:
  - When $y_p(t)$ is plugged in the nonhomog eq, we wish the left hand side would match the right hand side $36e^{3t} + 2e^{2t}$.
  - The $Be^{2t}$ part of the trial function satisfies the homog eq. That is, $(Be^{2t})'' - (Be^{2t})' - 2Be^{2t} = 0$. 
Example 6 (continued): $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

- Complementary solutions: $y_c = C_1e^{2t} + C_2e^{-t}$
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    That is, $(Be^{2t})'' - (Be^{2t})' - 2Be^{2t} = 0$.
  - In other words, when plugged in the nonhomog equation, this $Be^{2t}$ produces many terms, but the sum of those terms will simplify to zero!
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  - When \( y_p(t) \) is plugged in the nonhomog eq, we wish the left hand side would match the right hand side \( 36e^{3t} + 2e^{2t} \).
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  - In other words, when plugged in the nonhomog equation, this \( Be^{2t} \) produces many terms, but the sum of those terms will simplify to zero!
  - Thus, impossible to balance the two sides of the nonhomog equation.
Example 6 (continued): \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions: \( y_c = C_1e^{2t} + C_2e^{-t} \)
- The BAD trial function: \( y_p(t) = Ae^{3t} + Be^{2t} \).

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  - When \( y_p(t) \) is plugged in the nonhomog eq, we wish the left hand side would match the right hand side \( 36e^{3t} + 2e^{2t} \).
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  - In other words, when plugged in the nonhomog equation, this \( Be^{2t} \) produces many terms, but the sum of those terms will simplify to zero!
  - Thus, impossible to balance the two sides of the nonhomog equation.

- In short, the failure was due to the fact that \( y_p(t) \) has overlap(s) with \( y_c(t) \).
Example 6 (continued): \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions: \( y_c = C_1e^{2t} + C_2e^{-t} \)
- The BAD trial function: \( y_p(t) = Ae^{3t} + Be^{2t} \).
- **The Reason of the Failure:**
  - When \( y_p(t) \) is plugged in the nonhomog eq, we wish the left hand side would match the right hand side \( 36e^{3t} + 2e^{2t} \).
  - The \( Be^{2t} \) part of the trial function satisfies the homog eq. That is, \( (Be^{2t})'' - (Be^{2t})' - 2Be^{2t} = 0 \).
  - In other words, when plugged in the nonhomog equation, this \( Be^{2t} \) produces many terms, but the sum of those terms will simplify to zero!
  - Thus, impossible to balance the two sides of the nonhomog equation.

- In short, the failure was due to the fact that \( y_p(t) \) has overlap(s) with \( y_c(t) \).
- This kind of cases are called **resonance**.
  - The term \( 2e^{2t} \) in \( f(t) \) is called a **resonant term**.
Example 6 (continued): \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions: \( y_c(t) = C_1 e^{2t} + C_2 e^{-t} \)
- The **NAIVE** trial function: \( y_p(t) = Ae^{3t} + Be^{2t} \).
  This failed, since it has a term \( Be^{2t} \) overlapping \( y_c(t) \).
Example 6 (continued): \[ y'' - y' - 2y = 36e^{3t} + 2e^{2t}. \]

- Complementary solutions: \( y_c(t) = C_1 e^{2t} + C_2 e^{-t} \)
- The \textbf{NAIVE} trial function: \( y_p(t) = Ae^{3t} + Be^{2t}. \) This failed, since it has a term \( Be^{2t} \) overlapping \( y_c(t) \).
- The \textbf{CORRECT} trial function: \( y_p(t) = Ae^{3t} + Bte^{2t}. \)
Example 6 (continued): \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions: \( y_c(t) = C_1e^{2t} + C_2e^{-t} \)

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- **The Method of Correction:** In the naive trial function, multiply the bad term \( Be^{2t} \) by \( t^k \), where \( k \) is the smallest positive integer to ensure that \( y_p \) does not overlap \( y_c \).
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- Substitute \( y_p(t) \) in the nonhomog eq:

\[
36e^{3t} + 2e^{2t} = (Ae^{3t} + Bte^{2t})'' - (Ae^{3t} + Bte^{2t})' - 2(Ae^{3t} + Bte^{2t})
\]
\[
= (9Ae^{3t} + 4Be^{2t} + 4Bte^{2t}) - (3Ae^{3t} + Be^{2t} + 2Bte^{2t}) - 2(Ae^{3t} + Bte^{2t})
\]
\[
= 4Ae^{3t} + 3Be^{2t}
\]
Example 6 (continued): $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

- **Complementary solutions:** $y_c(t) = C_1e^{2t} + C_2e^{-t}$
- **The NAIVE trial function:** $y_p(t) = Ae^{3t} + Be^{2t}$.
  This failed, since it has a term $Be^{2t}$ overlapping $y_c(t)$.
- **The CORRECT trial function:** $y_p(t) = Ae^{3t} + Bte^{2t}$.
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  $$36e^{3t} + 2e^{2t} = (Ae^{3t} + Bte^{2t})'' - (Ae^{3t} + Bte^{2t})' - 2(Ae^{3t} + Bte^{2t})$$
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  $$- 2(Ae^{3t} + Bte^{2t})$$
  $$= 4Ae^{3t} + 3Be^{2t}$$
- **Compare the coefficients of the two sides:**
  $$\begin{align*}
 4A &= 36 \\
 3B &= 2 \\
\end{align*}$$
  $$\Rightarrow \begin{align*}
 4A &= 36 \\
 3B &= 2 \\
 3B &= 2/3 \\
 A &= 9 \\
 B &= 2/3 \\
\end{align*}$$
  $$\Rightarrow y_p(t) = 9e^{3t} + \frac{2}{3}te^{2t}$$
Example 6 (continued): \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

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- Substitute \( y_p(t) \) in the nonhomog eq:
  \[
  36e^{3t} + 2e^{2t} = (Ae^{3t} + Bte^{2t})'' - (Ae^{3t} + Bte^{2t})' - 2(Ae^{3t} + Bte^{2t})
  = (9Ae^{3t} + 4Be^{2t} + 4Bte^{2t}) - (3Ae^{3t} + Be^{2t} + 2Bte^{2t})
  - 2(Ae^{3t} + Bte^{2t})
  = 4Ae^{3t} + 3Be^{2t}
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- Compare the coefficients of the two sides:
  \[
  \begin{align*}
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  \end{align*}
  \Rightarrow
  \begin{align*}
  A &= 9 \\
  B &= 2/3
  \end{align*}
  \Rightarrow
  y_p(t) = 9e^{3t} + \frac{2}{3}te^{2t}
  \]
- The general solutions
  \[
  y(t) = y_p(t) + y_c(t) = 9e^{3t} + \frac{2}{3}te^{2t} + C_1e^{2t} + C_2e^{-t}
  \]
Example 7: Find a particular solution of
\[ y'' + 4y' + 4y = 9e^{4t} + (3 - 2t - t^2)e^{-2t} + \cos t. \]
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Complementary solutions:
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- The trial function:
  \[ y_p(t) = ae^{4t} + (b_0 + b_1 t + b_2 t^2)e^{-2t} + (A \cos t + B \sin t). \]
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- The trial function:
  \( y_p(t) = a e^{4t} + (b_0 + b_1 t + b_2 t^2)e^{-2t} + (A \cos t + B \sin t). \)
  This would fail, since the part \((b_0 + b_1 t)e^{-2t}\) overlaps \(y_c(t).\)
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  This would fail, since the part \((b_0 + b_1 t)e^{-2t}\) overlaps \(y_c(t)\).

- The modified trial function:
  \[ y_p(t) = ae^{4t} + t (b_0 + b_1 t + b_2 t^2)e^{-2t} + (A \cos t + B \sin t). \]
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\[y'' + 4y' + 4y = 9e^{4t} + (3 - 2t - t^2)e^{-2t} + \cos t.\]

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  \[y_p(t) = ae^{4t} + t^2(b_0 + b_1t + b_2t^2)e^{-2t} + (A \cos t + B \sin t).\]
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- Substitute this correct \(y_p(t)\) in the nonhomog eq:
  \[
  9e^{4t} + (3 - 2t - t^2)e^{-2t} + \cos t \quad = y_p'' + 4y_p' + 4y_p \quad = \cdots \cdots \]
  \[
  = 36ae^{4t} + (2b_0 + 6b_1t + 12b_2t^2)e^{-2t} + (3A + 4B)\cos t + (-4A + 3B)\sin t
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- Compare the coefficients of the two sides:
  \[ \begin{cases} 36a = 9 \\ 2b_0 = 3, \ 6b_1 = -2, \ 12b_2 = -1 \Rightarrow \begin{cases} a = \frac{1}{4} \\ b_0 = \frac{3}{2}, \ b_1 = -\frac{1}{3}, \ b_2 = -\frac{1}{12} \end{cases} \\ 3A + 4B = 1, \ -4A + 3B = 0 \Rightarrow \begin{cases} A = \frac{3}{25}, \ B = \frac{4}{25} \end{cases} \end{cases} \]
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  A &= \frac{3}{25}, \quad B &= \frac{4}{25}
  \end{align*}
  \]

- The particular solution
  \[ y_p(t) = \frac{1}{4} e^{4t} + t^2 \left( \frac{3}{2} - \frac{1}{3} t - \frac{1}{12} t^2 \right) e^{-2t} + \frac{3}{25} \cos t + \frac{4}{25} \sin t. \]
Example 8: Find a particular solution of
\[ y'' + 2y' + 10y = e^{-t} + \cos(3t) + e^{-t} \sin(3t). \]
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Complementary solutions: \( \lambda^2 + 2\lambda + 10 = 0 \Rightarrow \lambda_{1,2} = -1 \pm 3i \)
\[ y_c = C_1 e^{-t} \cos(3t) + C_2 e^{-t} \sin(3t) \]
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\( + [Ate^{-t} \cos(3t) + Bte^{-t} \sin(3t)] \)
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Substitute \(y_p(t)\) in the nonhomog eq and simplify:
\[e^{-t} + \cos(3t) + e^{-t} \sin(3t) = y_p'' + 2y_p' + 10y_p = \cdots \cdots\]
\[= 9ae^{-t} + (b_1 + 6b_2) \cos(3t) + (-6b_1 + b_2) \sin(3t)\]
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  \[ + 6Be^{-t} \cos(3t) - 6Ae^{-t} \sin(3t) \]

- Compare the coefficients of the two sides:
  \[ \begin{cases} 
  9a = 1 \\
  b_1 + 6b_2 = 1, \quad -6b_1 + b_2 = 0 \\
  6B = 0, \quad -6A = 1 
\end{cases} \quad \Rightarrow \quad \begin{cases} 
  a = 1/9 \\
  b_1 = 1/37, \quad b_2 = 6/37 \\
  A = -1/6, \quad B = 0 
\end{cases} \]
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  9a &= 1 \\
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  \end{align*} \]
  \[ \Rightarrow \begin{align*}
  a &= 1/9 \\
  b_1 &= 1/37, \quad b_2 &= 6/37 \\
  A &= -1/6, \quad B = 0
  \end{align*} \]

- The particular solution
  \[ y_p(t) = \frac{1}{10} e^{-2t} + \frac{1}{37} \cos(3t) + \frac{6}{37} \sin(3t) - \frac{1}{6} te^{-t} \cos(3t). \]
Summary of the Method of Undetermined Coefficients

Nonhomog Linear Equations: \( a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t) \)

How to set up the trial function?

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( y_p(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_N(t) ) (a polynomial of deg ( N ))</td>
<td>( A_0 + A_1 t + \cdots + A_N t^N )</td>
</tr>
<tr>
<td>( p_N(t)e^{rt} )</td>
<td>( (A_0 + A_1 t + \cdots + A_N t^N)e^{rt} )</td>
</tr>
<tr>
<td>( \left{ \begin{array}{l} p_N(t) \cos(\omega t) \ \text{and/or} \ p_N(t) \sin(\omega t) \end{array} \right} )</td>
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</tr>
</tbody>
</table>

In the case of resonance:
- First pick a naive trial function as in the above table.
- Then multiply the resonant term(s) by \( t^k \), where \( k \) is the smallest positive integer to ensure that \( y_p \) does not overlap \( y_c \).
Bad News
The method of undetermined coefficients does **NOOOOOOT** work, when the equation has variable coefficients:

\[
a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)
\]

*Example:* \((t - 1)y'' - ty' + y = e^{2t}\) cannot be solved by the m.u.c.
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The method of undetermined coefficients does NOOOOOOT work, when the equation has variable coefficients:

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Example: \((t - 1)y'' - ty' + y = e^{2t}\) cannot be solved by the m.u.c.

Even for the equations of constant coefficients:

\[ a_2y'' + a_1y' + a_0y = f(t), \]

the m.u.c. does NOOOOOOT always work.

It only works when \(f(t)\) is a linear combination of the functions that appear in the table of the last page.

Examples for which the m.u.c. fails:

\[ y'' + y = \tan t, \quad y'' + 2y' + y = e^{t^2}, \quad y'' - y = \frac{1}{1+t}, \quad \cdots \]
Bad News

- The method of undetermined coefficients does **NOT** work, when the equation has variable coefficients:
  \[ a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t) \]

  *Example:* \((t - 1)y'' - ty' + y = e^{2t}\) cannot be solved by the m.u.c.

- Even for the equations of constant coefficients:
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  It only works when \(f(t)\) is a linear combination of the functions that appear in the table of the last page.

  *Examples* for which the m.u.c. fails:
  \[ y'' + y = \tan t, \quad y'' + 2y' + y = e^{t^2}, \quad y'' - y = \frac{1}{1+t}, \quad \ldots \]

Good News

- There is a more general method, the *variation of parameters*, that can solve any nonhomog linear differential equation, as long as \(y_c\) has been provided/prepared.