Second Order Nonhomogeneous Linear Differential Equations with Constant Coefficients: 
the method of undetermined coefficients

Xu-Yan Chen
Second Order Nonhomogeneous Linear Differential Equations with Constant Coefficients:

\[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t), \]

where \( a_2 \neq 0, a_1, a_0 \) are constants, and \( f(t) \) is a given function (called the nonhomogeneous term).

General solution structure:

\[ y(t) = y_p(t) + y_c(t) \]

where \( y_p(t) \) is a particular solution of the nonhomog equation, and \( y_c(t) \) are solutions of the homogeneous equation:

\[ a_2 y_c''(t) + a_1 y_c'(t) + a_0 y_c(t) = 0. \]

The characteristic roots: \( a_2 \lambda^2 + a_1 \lambda + a_0 = 0 \)

\[ \Rightarrow \] The complementary solutions \( y_c(t) \).
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What is this note about? The Method of Undetermined Coefficients:
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What is this note about? The Method of Undetermined Coefficients: a method of finding \( y_p(t) \), when the nonhomog term \( f(t) \) belongs a simple class.

Main Idea: Set up a trial function \( y_p(t) \), by copying the function form of \( f(t) \).
Example 1: Solve $3y'' + y' - 2y = 10e^{4t}$, $y(0) = -1$, $y'(0) = 3$. 
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  $3y''_c + y'_c - 2y_c = 0$  \hspace{1cm} (the corresponding homog eq)
  
  $3\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = \frac{2}{3} \Rightarrow y_c = C_1 e^{-t} + C_2 e^{\frac{2}{3}t}$
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- To find $y_p(t)$, set the trial function

  $y_p(t) = ae^{4t}$ (form copied from $f(t) = 10e^{4t}$)

where $a$ is the undetermined coefficient.
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  where $a$ is the undetermined coefficient.

- Substitute $y_p(t)$ in the nonhomog eq:
  
  $3(ae^{4t})'' + (ae^{4t})' - 2ae^{4t} = 10e^{4t}$

  $= 3(16ae^{4t}) + (4ae^{4t}) - 2ae^{4t}$

  $= 50ae^{4t}$
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- Find complementary solutions \( y_c(t) \):
  
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  3y''_c + y'_c - 2y_c = 0 \quad \text{(the corresponding homog eq)}
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- To find \( y_p(t) \), set the trial function
  
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- Compare the coefficients of the two sides:
  
  \[
  50a = 10 \Rightarrow a = \frac{1}{5}
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Example 1 (continued): Solve
\[ 3y'' + y' - 2y = 10e^{4t}, \quad y(0) = -1, y'(0) = 3. \]

Combine \( y_c \) and \( y_p \) to get

Gen Sols of Nonhomg Eq:  
\[ y(t) = \frac{1}{5}e^{4t} + C_1 e^{-t} + C_2 e^{\frac{2}{3}t}. \]
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- Use initial conditions:

\[ y(0) = -1 \quad \Rightarrow \quad \frac{1}{5} + C_1 + C_2 = -1 \]

\[ y'(t) = \frac{4}{5}e^{4t} - C_1 e^{-t} + \frac{2}{3}C_2 e^{\frac{2}{3}t}, \quad y'(0) = 3 \quad \Rightarrow \quad \frac{4}{5} - C_1 + \frac{2}{3}C_2 = 3 \]
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Solve this:

\[
\begin{cases}
C_1 = -\frac{9}{5} \\
C_2 = \frac{3}{5}
\end{cases}
\]
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\begin{cases}
C_1 = -\frac{9}{5} \\
C_2 = \frac{3}{5}
\end{cases}
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- The solution of the initial value problem:

\[
y(t) = \frac{1}{5}e^{4t} - \frac{9}{5}e^{-t} + \frac{3}{5}e^{\frac{2}{3}t}.
\]
Nonhomogeneous Linear Equations:

\[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t), \]

Towards the Rules of Setting Up the Trial Function:

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( y_p(t) )</th>
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<tr>
<td>( ke^{rt} )</td>
<td>( Ae^{rt} )</td>
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- Complementary solutions $y_c(t)$:
  
  
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- To find $y_p(t)$, set the trial function
  
  $$y_p(t) = Ate^{-2t}.$$
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- Substitute $y_p(t)$ in the nonhomog eq:
  
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  $$= 3(-4Ae^{-2t} + 4Ate^{-2t}) + (Ae^{-2t} - 2Ate^{-2t}) - 2Ate^{-2t}$$
  
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  \]

- Compare the coefficients of the two sides:
  
  \[
  \begin{cases}
  -11A = 0 \\
  8A = -8
  \end{cases} \quad \Rightarrow \quad \text{Impossible!}
  \]
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- To find $y_p(t)$, set the trial function
  
  $y_p(t) = Ate^{-2t}$.

- Substitute $y_p(t)$ in the nonhomog eq:
  
  $-8te^{-2t} = 3(At^{-2t})'' + (At^{-2t})' - 2Ate^{-2t}$
  
  $= 3(-4Ae^{-2t} + 4Ate^{-2t}) + (Ae^{-2t} - 2Ate^{-2t}) - 2Ate^{-2t}$
  
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- Compare the coefficients of the two sides:
  
  $\begin{cases} -11A = 0 \\ 8A = -8 \end{cases} \Rightarrow \text{Impossible!}$

The choice of the trial function $y_p(t) = Ate^{-2t}$ was {WRONG}!
Example 2 (continued): Solve $3y'' + y' - 2y = -8te^{-2t}$.

- The correct point of view:

  $$f(t) = -8te^{-2t} = (\text{a polynomial of degree one})e^{-2t}.$$
Example 2 (continued): Solve $3y'' + y' - 2y = -8te^{-2t}$.

- The correct point of view:
  
  $$f(t) = -8te^{-2t} = (\text{a polynomial of degree one})e^{-2t}.$$ 

- The correct trial function:
  
  $$y_p(t) = (A + Bt)e^{-2t}.$$
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- The correct trial function:
  
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- Substitute $y_p(t)$ in the nonhomog eq:
  
  \[
  -8te^{-2t} = 3[(A + Bt)e^{-2t}]'' + [(A + Bt)e^{-2t}]' - 2(A + Bt)e^{-2t}
  = 3(4A - 4B + 4Bt)e^{-2t} + (-2A + B - 2Bt)e^{-2t} + (-2A - 2Bt)e^{-2t}
  = (8A - 11B)e^{-2t} + 8Bte^{-2t}
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  $+ (-2A - 2Bt)e^{-2t}$
  
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- Compare the coefficients of the two sides:
  
  \[ \begin{aligned}
  8A - 11B &= 0 \\
  8B &= -8
  \end{aligned} \]

  \[ \Rightarrow \begin{aligned}
  A &= -\frac{11}{8} \\
  B &= -1
  \end{aligned} \]

  \[ \Rightarrow y_p(t) = \left(-\frac{11}{8} - t\right)e^{-2t} \]
Example 2 (continued): Solve $3y'' + y' - 2y = -8te^{-2t}$.

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  \]

- The General Solutions of the Nonhomogeneous Equation:
  \[ y(t) = y_p(t) + y_c(t) = \left(-\frac{11}{8} - t\right)e^{-2t} + C_1 e^{-t} + C_2 e^{2t}. \]
Example 3: Solve $3y'' + y' - 2y = -12t^2$. 
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Complementary solutions $y_c(t)$:

$3\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 2/3 \Rightarrow y_c = C_1 e^{-t} + C_2 e^{2/3 t}$
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- To find $y_p(t)$, set the trial function
  
  $$y_p(t) = At^2.$$
Example 3: Solve $3y'' + y' - 2y = -12t^2$.

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▶ To find $y_p(t)$, set the trial function

$$y_p(t) = At^2.$$ This does not work!
Example 3: Solve $3y'' + y' - 2y = -12t^2$.

- Complementary solutions $y_c(t)$:
  
  $3\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = \frac{2}{3} \Rightarrow y_c = C_1 e^{-t} + C_2 e^{\frac{2}{3}t}$

- To find $y_p(t)$, set the trial function
  
  $y_p(t) = A t^2$. This does not work!

- The correct trial function:
  
  $y_p(t) = A + Bt + Ct^2$. 
**Example 3:** Solve $3y'' + y' - 2y = -12t^2$.

- Complementary solutions $y_c(t)$:
  
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  3\lambda^2 + \lambda - 2 = 0 \implies \lambda_1 = -1, \lambda_2 = 2/3 \implies y_c = C_1 e^{-t} + C_2 e^{2/3t}
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- To find $y_p(t)$, set the trial function
  
  \[
  y_p(t) = At^2. \quad \text{This does not work!}
  \]

- The correct trial function:
  \[
  y_p(t) = A + Bt + Ct^2.
  \]

- Substitute $y_p(t)$ in the nonhomog eq:
  \[
  -12t^2 = 3(2C) + (B + 2Ct) - 2(A + Bt + Ct^2)
  \]
  \[
  = (-2A + B + 6C) + (-2B + 2C)t - 2Ct^2
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- Substitute $y_p(t)$ in the nonhomog eq:
  
  $$-12t^2 = 3(2C) + (B + 2 Ct) - 2(A + Bt + Ct^2)$$

  $$= (-2A + B + 6C) + (-2B + 2C)t - 2Ct^2$$

- Compare the coefficients of the two sides:

  $$\begin{cases} 
  -2A + B + 6C = 0 \\
  -2B + 2C = 0 \\
  -2C = -12
  \end{cases} \Rightarrow \begin{cases} 
  A = 21 \\
  B = 6 \Rightarrow y_p(t) = 21 + 6t + 6t^2 \\
  C = 6
  \end{cases}$$
Example 3: Solve $3y'' + y' - 2y = -12t^2$.

- Complementary solutions $y_c(t)$:
  
  $$3\lambda^2 + \lambda - 2 = 0 \implies \lambda_1 = -1, \lambda_2 = \frac{2}{3} \implies y_c = C_1 e^{-t} + C_2 e^{\frac{2}{3}t}$$

- To find $y_p(t)$, set the trial function
  
  $$y_p(t) = At^2.$$  
  This does not work!

- The correct trial function:
  
  $$y_p(t) = A + Bt + Ct^2.$$ 

- Substitute $y_p(t)$ in the nonhomogeneous equation:
  
  $$-12t^2 = 3(2C) + (B + 2Ct) - 2(A + Bt + Ct^2)$$
  
  $$= (-2A + B + 6C) + (-2B + 2C)t - 2Ct^2$$

- Compare the coefficients of the two sides:
  
  $$\begin{cases} 
  -2A + B + 6C = 0 \\
  -2B + 2C = 0 \quad \Rightarrow \quad A = 21 \\
  -2C = -12 \quad \Rightarrow \quad B = 6 \quad \Rightarrow \quad y_p(t) = 21 + 6t + 6t^2 \\
  \end{cases}$$

- The General Solutions of the Nonhomogeneous Equation:
  
  $$y(t) = y_p(t) + y_c(t) = 21 + 6t + 6t^2 + C_1 e^{-t} + C_2 e^{\frac{2}{3}t}.$$
Nonhomogeneous Linear Equations:

\[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t), \]

Towards the Rules of Setting Up the Trial Function:

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(to be continued)
Example 4: Find a particular solution of $3y'' + y' - 2y = 5 \cos(2t)$. 
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- Complementary solutions $y_c(t)$:

  
  $3\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 2/3 \Rightarrow y_c = C_1e^{-t} + C_2e^{2/3t}$
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- To find $y_p(t)$, set the trial function
  
  $y_p(t) = A \cos(2t)$. 
Example 4: Find a particular solution of $3y'' + y' - 2y = 5 \cos(2t)$.

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- To find $y_p(t)$, set the trial function
  
  $y_p(t) = A \cos(2t)$.

- Substitute $y_p(t)$ in the nonhomog eq:

  $5 \cos(2t) = 3[A \cos(2t)]'' + [A \cos(2t)]' - 2A \cos(2t)$
  
  $= 3(-4A \cos(2t)) - 2A \sin(2t) - 2A \cos(2t)$
  
  $= -14A \cos(2t) - 2A \sin(2t)$
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- To find $y_p(t)$, set the trial function
  
  $$y_p(t) = A \cos(2t)$$

- Substitute $y_p(t)$ in the nonhomog eq:
  
  $$5 \cos(2t) = 3[A \cos(2t)]'' + [A \cos(2t)]' - 2A \cos(2t)$$
  
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  $$= -14A \cos(2t) - 2A \sin(2t)$$

- Compare the coefficients of the two sides:
  
  $$\left\{ \begin{array}{l}
  -14A = 5 \\
  -2A = 0
  \end{array} \right. \Rightarrow \text{Impossible!}$$
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  $$y_p(t) = A \cos(2t).$$

- Substitute $y_p(t)$ in the nonhomog eq:

  
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- Compare the coefficients of the two sides:

  $$\left\{ \begin{array}{l}
  -14A = 5 \\
  -2A = 0 \\
  \end{array} \right. \Rightarrow \text{Impossible!}$$

  The choice of the trial function $y_p(t) = A \cos(2t)$ was **WRONG!**
Example 4 (continued): Find a particular solution of
\[ 3y'' + y' - 2y = 5 \cos(2t). \]

The correct trial function:
\[ y_p(t) = A \cos(2t) + B \sin(2t). \]
Example 4 (continued): Find a particular solution of
\( 3y'' + y' - 2y = 5 \cos(2t) \).

The correct trial function:
\[
y_p(t) = A \cos(2t) + B \sin(2t).
\]

Substitute \( y_p(t) \) in the nonhomog eq:
\[
5 \cos(2t) = 3[A \cos(2t) + B \sin(2t)]'' + [A \cos(2t) + B \sin(2t)]'
- 2[A \cos(2t) + B \sin(2t)]
= 3[-4A \cos(2t) - 4B \sin(2t)] + [-2A \sin(2t) + 2B \cos(2t)]
- 2[A \cos(2t) + B \sin(2t)]
= (-14A + 2B) \cos(2t) + (-2A - 14B) \sin(2t).
\]
Example 4 (continued): Find a particular solution of 
\[3y'' + y' - 2y = 5\cos(2t).\]

- The correct trial function:
  \[y_p(t) = A\cos(2t) + B\sin(2t).\]

- Substitute \(y_p(t)\) in the nonhomog eq:
  
  \[
  5\cos(2t) = 3[A\cos(2t) + B\sin(2t)]'' + [A\cos(2t) + B\sin(2t)]'
  - 2[A\cos(2t) + B\sin(2t)]
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  - 2[A\cos(2t) + B\sin(2t)]
  = (-14A + 2B)\cos(2t) + (-2A - 14B)\sin(2t).
  \]

- Compare the coefficients of the two sides:
  \[
  \begin{cases}
  -14A + 2B = 5 \\
  -2A - 14B = 0
  \end{cases}
  \Rightarrow
  \begin{cases}
  A = -\frac{7}{20} \\
  B = \frac{1}{20}
  \end{cases}
  \]
Example 4 (continued): Find a particular solution of \(3y'' + y' - 2y = 5 \cos(2t)\).

- The correct trial function:
  \[y_p(t) = A \cos(2t) + B \sin(2t).\]

- Substitute \(y_p(t)\) in the nonhomog eq:
  \[
  5 \cos(2t) = 3[A \cos(2t) + B \sin(2t)]'' + [A \cos(2t) + B \sin(2t)]'
  - 2[A \cos(2t) + B \sin(2t)]
  = 3[-4A \cos(2t) - 4B \sin(2t)] + [-2A \sin(2t) + 2B \cos(2t)]
  - 2[A \cos(2t) + B \sin(2t)]
  = (-14A + 2B) \cos(2t) + (-2A - 14B) \sin(2t).
  \]

- Compare the coefficients of the two sides:
  \[
  \begin{align*}
  -14A + 2B &= 5 \\
  -2A - 14B &= 0 \\
  \end{align*}
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  \begin{align*}
  A &= -\frac{7}{20} \\
  B &= \frac{1}{20} \\
  \end{align*}
  \]

- A Particular Solution of the Nonhomogeneous Equation:
  \[y_p(t) = -\frac{7}{20} \cos(2t) + \frac{1}{20} \sin(2t).\]
Nonhomogeneous Linear Equations:

\[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t) \]

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  \text{and/or} \\
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(to be continued)
Example 5: Find a particular solution of

$$3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5\cos(2t) + 17e^{-t}\cos t + 34e^{-t}\sin t$$
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We give two methods.
Example 5: Find a particular solution of

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We give two methods.

Method 1:

Solve \[3y_1'' + y'_1 - 2y_1 = 10e^{4t}\] to get a particular solution \(y_1(t)\).

Solve \[3y_2'' + y'_2 - 2y_2 = -8te^{-2t}\] to get a particular solution \(y_2(t)\).

Solve \[3y_3'' + y'_3 - 2y_3 = -12t^2\] to get a particular solution \(y_3(t)\).

Solve \[3y_4'' + y'_4 - 2y_4 = 5\cos(2t)\] to get a particular solution \(y_4(t)\).

Solve \[3y_5'' + y'_5 - 2y_5 = 17e^{-t}\cos t + 34e^{-t}\sin t\] to get a particular solution \(y_5(t)\). (Set \(y_5(t) = Ae^{-t}\cos t + Be^{-t}\sin t\))
**Example 5:** Find a particular solution of

\[3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5 \cos(2t) + 17e^{-t} \cos t + 34e^{-t} \sin t\]

We give two methods.

**Method 1:**

Solve \(3y_1'' + y_1' - 2y_1 = 10e^{4t}\) to get a particular solution \(y_1(t)\).

Solve \(3y_2'' + y_2' - 2y_2 = -8te^{-2t}\) to get a particular solution \(y_2(t)\).

Solve \(3y_3'' + y_3' - 2y_3 = -12t^2\) to get a particular solution \(y_3(t)\).

Solve \(3y_4'' + y_4' - 2y_4 = 5 \cos(2t)\) to get a particular solution \(y_4(t)\).

Solve \(3y_5'' + y_5' - 2y_5 = 17e^{-t} \cos t + 34e^{-t} \sin t\) to get a particular solution \(y_5(t)\).

(Set \(y_5(t) = Ae^{-t} \cos t + Be^{-t} \sin t\))

A particular solution to the original equation:

\[
y_p(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t) + y_5(t)
= \frac{1}{5} e^{2t} + \left(-\frac{11}{8} - t\right) e^{-2t} + 21 + 6t + 6t^2
- \frac{7}{20} \cos(2t) + \frac{1}{20} \sin(2t) + \frac{7}{2} e^{-t} \cos t - \frac{11}{2} e^{-t} \sin t.
\]
Example 5 (continued): Find a particular solution of

$$3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5 \cos(2t) + 17e^{-t} \cos t + 34e^{-t} \sin t$$

Method 2:

- Set a **BIIIIIG** trial function:

$$y_p(t) = A_0 e^{4t} + (A_1 + A_2 t) e^{-2t} + (A_3 + A_4 t + A_5 t^2)$$

$$+ [A_6 \cos(2t) + A_7 \sin(2t)] + (A_8 e^{-t} \cos t + A_9 e^{-t} \sin t),$$

with undetermined coefficients $A_0, A_1, \cdots, A_9$. 
Example 5 (continued): Find a particular solution of

\[3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5\cos(2t) + 17e^{-t}\cos t + 34e^{-t}\sin t\]

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- Substitute this in the original equation.
Example 5 (continued): Find a particular solution of

\[ 3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5 \cos(2t) + 17e^{-t} \cos t + 34e^{-t} \sin t \]

Method 2:

- Set a **BIIIIIG** trial function:

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  \[ + [A_6 \cos(2t) + A_7 \sin(2t)] + (A_8 e^{-t} \cos t + A_9 e^{-t} \sin t), \]
  
  with undetermined coefficients \( A_0, A_1, \cdots, A_9 \).

- Substitute this in the original equation.

- Compare the coefficients of the two sides

  \[ \Rightarrow \] Linear equations for \( A_0, A_1, \cdots A_9 \).
Example 5 (continued): Find a particular solution of
\[3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5\cos(2t) + 17e^{-t}\cos t + 34e^{-t}\sin t\]

Method 2:

- Set a BIG trial function:
  \[y_p(t) = A_0e^{4t} + (A_1 + A_2t)e^{-2t} + (A_3 + A_4t + A_5t^2)\]
  \[+ [A_6\cos(2t) + A_7\sin(2t)] + (A_8e^{-t}\cos t + A_9e^{-t}\sin t),\]
  with undetermined coefficients \(A_0, A_1, \cdots, A_9\).

- Substitute this in the original equation.

- Compare the coefficients of the two sides
  \[\Rightarrow\] Linear equations for \(A_0, A_1, \cdots A_9\).

- Solve \(A_0, A_1, \cdots A_9\).
Example 5 (continued): Find a particular solution of 

\[ 3y'' + y' - 2y = 10e^{4t} - 8te^{-2t} - 12t^2 + 5 \cos(2t) + 17e^{-t} \cos t + 34e^{-t} \sin t \]

Method 2:

▷ Set a **BIIIIG** trial function:

\[
y_p(t) = A_0 e^{4t} + (A_1 + A_2 t)e^{-2t} + (A_3 + A_4 t + A_5 t^2) \\
+ [A_6 \cos(2t) + A_7 \sin(2t)] + (A_8 e^{-t} \cos t + A_9 e^{-t} \sin t),
\]

with undetermined coefficients \( A_0, A_1, \cdots, A_9 \).

▷ Substitute this in the original equation.

▷ Compare the coefficients of the two sides  
  \( \Rightarrow \) Linear equations for \( A_0, A_1, \cdots A_9 \).

▷ Solve \( A_0, A_1, \cdots A_9 \).

▷ Finally obtain the particular solution 

\[
y_p(t) = \frac{1}{5} e^{2t} + \left(-\frac{11}{8} - t \right) e^{-2t} + 21 + 6t + 6t^2 \\
- \frac{7}{20} \cos(2t) + \frac{1}{20} \sin(2t) + \frac{7}{2} e^{-t} \cos t - \frac{11}{2} e^{-t} \sin t.
\]

(Computational details skipped here.)
Example 6: Find general solutions of $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$. 
Example 6: Find general solutions of $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

Complementary solutions $y_c(t)$:

$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -1 \Rightarrow y_c = C_1 e^{2t} + C_2 e^{-t}$
Example 6: Find general solutions of \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions \( y_c(t) \):
  \[
  \lambda^2 - \lambda - 2 = 0 \implies \lambda_1 = 2, \lambda_2 = -1 \implies y_c = C_1 e^{2t} + C_2 e^{-t}
  \]

- To find \( y_p(t) \), set the trial function
  \[
  y_p(t) = Ae^{3t} + Be^{2t}.
  \]
Example 6: Find general solutions of \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions \( y_c(t) \):
  \[
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  \]

- To find \( y_p(t) \), set the trial function
  \[
  y_p(t) = Ae^{3t} + Be^{2t}.
  \]

- Substitute \( y_p(t) \) in the nonhomog eq:
  \[
  36e^{3t} + 2e^{2t} = (Ae^{3t} + Be^{2t})'' - (Ae^{3t} + Be^{2t})' - 2(Ae^{3t} + Be^{2t})
  \]
  \[
  = (9Ae^{3t} + 4Be^{2t}) - (3Ae^{3t} + 2Be^{2t}) - 2(Ae^{3t} + Be^{2t})
  \]
  \[
  = 4Ae^{3t}
  \]
Example 6: Find general solutions of \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions \( y_c(t) \):
  \[ \lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -1 \Rightarrow y_c = C_1 e^{2t} + C_2 e^{-t} \]

- To find \( y_p(t) \), set the trial function
  \[ y_p(t) = Ae^{3t} + Be^{2t}. \]

- Substitute \( y_p(t) \) in the nonhomog eq:
  \[
  36e^{3t} + 2e^{2t} = (Ae^{3t} + Be^{2t})'' - (Ae^{3t} + Be^{2t})' - 2(Ae^{3t} + Be^{2t})
  = (9Ae^{3t} + 4Be^{2t}) - (3Ae^{3t} + 2Be^{2t}) - 2(Ae^{3t} + Be^{2t})
  = 4Ae^{3t}
  \]

- Compare the coefficients of the two sides:
  \[
  \begin{cases} 
  4A = 36 \\
  0 = 2 \\
  \end{cases} \Rightarrow \text{Impossible!}
  \]
Example 6: Find general solutions of \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions \( y_c(t) \):
  \[ \lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -1 \Rightarrow y_c = C_1 e^{2t} + C_2 e^{-t} \]

- To find \( y_p(t) \), set the trial function \( y_p(t) = Ae^{3t} + Be^{2t} \).

- Substitute \( y_p(t) \) in the nonhomog eq:
  \[
  36e^{3t} + 2e^{2t} = (Ae^{3t} + Be^{2t})'' - (Ae^{3t} + Be^{2t})' - 2(Ae^{3t} + Be^{2t})
  = (9Ae^{3t} + 4Be^{2t}) - (3Ae^{3t} + 2Be^{2t}) - 2(Ae^{3t} + Be^{2t})
  = 4Ae^{3t}
  \]

- Compare the coefficients of the two sides:
  \[
  \begin{cases}
    4A = 36 \\
    0 = 2
  \end{cases} \Rightarrow \text{Impossible!}
  \]

The trial function \( y_p(t) = Ae^{3t} + Be^{2t} \) was \textbf{BAD}!
Example 6 (continued): \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions: \( y_c = C_1e^{2t} + C_2e^{-t} \)
- The **BAD** trial function: \( y_p(t) = Ae^{3t} + Be^{2t} \).
Example 6 (continued): \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions: \( y_c = C_1e^{2t} + C_2e^{-t} \)
- The BAD trial function: \( y_p(t) = Ae^{3t} + Be^{2t} \).
- The Reason of the Failure:
Example 6 (continued): \[ y'' - y' - 2y = 36e^{3t} + 2e^{2t}. \]

- Complementary solutions: \[ y_c = C_1 e^{2t} + C_2 e^{-t} \]
- The BAD trial function: \[ y_p(t) = Ae^{3t} + Be^{2t}. \]
- The Reason of the Failure:
  - When \( y_p(t) \) is plugged in the nonhomog eq, we wish the left hand side would match the right hand side \( 36e^{3t} + 2e^{2t} \).
Example 6 (continued): \[ y'' - y' - 2y = 36e^{3t} + 2e^{2t}. \]

- Complementary solutions: \[ y_c = C_1e^{2t} + C_2e^{-t} \]
- The **BAD** trial function: \[ y_p(t) = Ae^{3t} + Be^{2t}. \]
- The **Reason of the Failure**:  
  - When \( y_p(t) \) is plugged in the nonhomog eq, we wish the left hand side would match the right hand side \( 36e^{3t} + 2e^{2t} \).
  - The \( Be^{2t} \) part of the trial function satisfies the homog eq. That is, \( (Be^{2t})'' - (Be^{2t})' - 2Be^{2t} = 0. \)
Example 6 (continued):  \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions:  \( y_c = C_1e^{2t} + C_2e^{-t} \)
- The \textbf{BAD} trial function:  \( y_p(t) = Ae^{3t} + Be^{2t} \).
- The \textbf{Reason of the Failure}:
  - When  \( y_p(t) \) is plugged in the nonhomog eq, we wish the left hand side would match the right hand side  \( 36e^{3t} + 2e^{2t} \).
  - The  \( Be^{2t} \) part of the trial function satisfies the homog eq. That is,  \( (Be^{2t})'' - (Be^{2t})' - 2Be^{2t} = 0 \).
  - In other words, when plugged in the nonhomog equation, this  \( Be^{2t} \) produces many terms, but the sum of those terms will simplify to zero!
Example 6 (continued):  \( y'' - y' - 2y = 36e^{3t} + 2e^{2t} \).

- Complementary solutions: \( y_c = C_1 e^{2t} + C_2 e^{-t} \)

- The **BAD** trial function: \( y_p(t) = Ae^{3t} + Be^{2t} \).

- **The Reason of the Failure:**
  - When \( y_p(t) \) is plugged in the nonhomog eq, we wish the left hand side would match the right hand side \( 36e^{3t} + 2e^{2t} \).
  - The \( Be^{2t} \) part of the trial function satisfies the homog eq. That is, \((Be^{2t})'' - (Be^{2t})' - 2Be^{2t} = 0\).
  - In other words, when plugged in the nonhomog equation, this \( Be^{2t} \) produces many terms, but the sum of those terms will simplify to zero!
  - Thus, impossible to balance the two sides of the nonhomog equation.
Example 6 (continued): $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

- Complementary solutions: $y_c = C_1 e^{2t} + C_2 e^{-t}$
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- In short, the failure was due to the fact that $y_p(t)$ has overlap(s) with $y_c(t)$.
Complementary solutions: $y_c = C_1 e^{2t} + C_2 e^{-t}$

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In short, the failure was due to the fact that $y_p(t)$ has overlap(s) with $y_c(t)$.

This kind of cases are called resonance.

The term $2e^{2t}$ in $f(t)$ is called a resonant term.
Example 6 (continued):  
\[ y'' - y' - 2y = 36e^{3t} + 2e^{2t}. \]

- Complementary solutions:  
  \[ y_c(t) = C_1e^{2t} + C_2e^{-t}. \]

- The **NAIVE** trial function:  
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  \[
  36e^{3t} + 2e^{2t} = (Ae^{3t} + Bte^{2t})'' - (Ae^{3t} + Bte^{2t})' - 2(Ae^{3t} + Bte^{2t})
  \]

  \[
  = (9Ae^{3t} + 4Be^{2t} + 4Bte^{2t}) - (3Ae^{3t} + Be^{2t} + 2Bte^{2t})
  \]

  \[
  -2(Ae^{3t} + Bte^{2t})
  \]

  \[
  = 4Ae^{3t} + 3Be^{2t}
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Example 6 (continued): $y'' - y' - 2y = 36e^{3t} + 2e^{2t}$.

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- Compare the coefficients of the two sides:
  
  $$\begin{cases} 4A = 36 \\ 3B = 2 \end{cases} \Rightarrow \begin{cases} A = 9 \\ B = 2/3 \end{cases} \Rightarrow y_p(t) = 9e^{3t} + \frac{2}{3}te^{2t}$$
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  \end{align*}
  \Rightarrow \begin{align*}
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  \end{align*}
  \Rightarrow y_p(t) = 9e^{3t} + \frac{2}{3}te^{2t}

- The general solutions
  
  \[
  y(t) = y_p(t) + y_c(t) = 9e^{3t} + \frac{2}{3}te^{2t} + C_1e^{2t} + C_2e^{-t}
  
\]
Example 7: Find a particular solution of
\[ y'' + 4y' + 4y = 9e^{4t} + (3 - 2t - t^2)e^{-2t} + \cos t. \]
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- The modified trial function:
  \[ y_p(t) = ae^{4t} + t (b_0 + b_1t + b_2t^2)e^{-2t} + (A \cos t + B \sin t). \]
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- Substitute this correct \(y_p(t)\) in the nonhomog eq:
  \[ 9e^{4t} + (3 - 2t - t^2)e^{-2t} + \cos t = y_p'' + 4y_p' + 4y_p = \cdots \cdots \]
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  36a &= 9 \\
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  \end{align*}
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  \[
  \begin{align*}
  a &= \frac{1}{4} \\
  b_0 &= \frac{3}{2}, \quad b_1 &= -\frac{1}{3}, \quad b_2 = -\frac{1}{12} \\
  A &= \frac{3}{25}, \quad B = \frac{4}{25} \end{align*}
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    A = \frac{3}{25}, \ B = \frac{4}{25}
  \end{cases}
  \]

- The particular solution
  \[ y_p(t) = \frac{1}{4}e^{4t} + t^2 \left( \frac{3}{2} - \frac{1}{3} t - \frac{1}{12} t^2 \right) e^{-2t} + \frac{3}{25} \cos t + \frac{4}{25} \sin t. \]
Example 8: Find a particular solution of
\[ y'' + 2y' + 10y = e^{-t} + \cos(3t) + e^{-t} \sin(3t). \]
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- The trial function: 
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- Substitute \( y_p(t) \) in the nonhomog eq and simplify:
  
  \[ e^{-t} + \cos(3t) + e^{-t} \sin(3t) \]
  \[ = y''_p + 2y'_p + 10y_p \]
  \[ = 9ae^{-t} + (b_1 + 6b_2) \cos(3t) + (-6b_1 + b_2) \sin(3t) \]
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- Compare the coefficients of the two sides:
  \[
  \begin{align*}
  9a &= 1 \\
  b_1 + 6b_2 &= 1, \quad -6b_1 + b_2 &= 0 \\
  6B &= 0, \quad -6A &= 1
  \end{align*}
  \]  
  \[
  \begin{align*}
  a &= 1/9 \\
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\end{cases} \Rightarrow \begin{cases} 
  a &= 1/9 \\
  b_1 &= 1/37, \quad b_2 &= 6/37 \\
  A &= -1/6, \quad B &= 0 
\end{cases} \]

- The particular solution 
  \[ y_p(t) = \frac{1}{10} e^{-2t} + \frac{1}{37} \cos(3t) + \frac{6}{37} \sin(3t) - \frac{1}{6} te^{-t} \cos(3t). \]
Summary of the Method of Undetermined Coefficients

Nonhomog Linear Equations: \[ a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t) \]

How to set up the trial function?

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( y_p(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_N(t) ) (a polynomial of deg ( N ))</td>
<td>( A_0 + A_1 t + \cdots + A_N t^N )</td>
</tr>
<tr>
<td>( p_N(t)e^{rt} )</td>
<td>((A_0 + A_1 t + \cdots + A_N t^N)e^{rt})</td>
</tr>
<tr>
<td>( \left{ \begin{array}{l} p_N(t) \cos(\omega t) \ \text{and/or} \quad p_N(t) \sin(\omega t) \end{array} \right} )</td>
<td>( (A_0 + A_1 t + \cdots + A_N t^N) \cos(\omega t) + (B_0 + B_1 t + \cdots + B_N t^N) \sin(\omega t) )</td>
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</tr>
</tbody>
</table>

In the case of resonance:
- First pick a naive trial function as in the above table.
- Then multiply the resonant term(s) by \( t^k \), where \( k \) is the smallest positive integer to ensure that \( y_p \) does not overlap \( y_c \).
Bad News
Bad News

- The method of undetermined coefficients does **NOT** work, when the equation has variable coefficients:

\[ a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t) \]

*Example:* \((t - 1)y'' - ty' + y = e^{2t}\) cannot be solved by the m.u.c.
Bad News

- The method of undetermined coefficients does NOOOOOOT work, when the equation has variable coefficients:
  \[ a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t) \]

  Example: \((t-1)y'' - ty' + y = e^{2t}\) cannot be solved by the m.u.c.

- Even for the equations of constant coefficients:
  \[ a_2y'' + a_1y' + a_0y = f(t), \]
  the m.u.c. does NOOOOOOT always work.
  It only works when \(f(t)\) is a linear combination of the functions that appear in the table of the last page.

  Examples for which the m.u.c. fails:
  \[ y'' + y = \tan t, \quad y'' + 2y' + y = e^{t^2}, \quad y'' - y = \frac{1}{1+t}, \quad \cdots \]
Bad News

- The method of undetermined coefficients does **NOT** work, when the equation has variable coefficients:
  \[ a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t) \]

  *Example:* \((t - 1)y'' - ty' + y = e^{2t}\) cannot be solved by the m.u.c.

- Even for the equations of constant coefficients:
  \[ a_2y'' + a_1y' + a_0y = f(t), \]

  the m.u.c. does **NOT** always work.

  It only works when \(f(t)\) is a linear combination of the functions that appear in the table of the last page.

  *Examples* for which the m.u.c. fails:
  \[ y'' + y = \tan t, \quad y'' + 2y' + y = e^{t^2}, \quad y'' - y = \frac{1}{1+t}, \quad \cdots \]

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Good News

- There is a more general method, the *variation of parameters*, that can solve any nonhomog linear differential equation, as long as \(y_c\) has been provided/prepared.