Matrix Exponential. Fundamental Matrix Solution.

Objective: Solve

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$$

with an $n \times n$ constant coefficient matrix A.

Here, the unknown is the vector function $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$.

General Solution Formula in Matrix Exponential Form:

$$\vec{\mathbf{x}}(t) = e^{tA}\vec{\mathbf{C}} = e^{tA} \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix},$$

where $C_1, \cdots, C_n$ are arbitrary constants.

The solution of the initial value problem

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$$

is given by

$$\vec{\mathbf{x}}(t) = e^{(t-t_0)A}\vec{\mathbf{x}}_0.$$

Definition (Matrix Exponential): For a square matrix A,

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots.$$

Evaluation of Matrix Exponential in the Diagonalizable Case: Suppose that A is diagonalizable; that is, there are an invertible matrix P and a diagonal matrix $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ such that $A = PDP^{-1}$. In this case, we have

$$e^{tA} = Pe^{tD}P^{-1} = P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}.$$

EXAMPLE 1. Let $A = \begin{bmatrix} 6 & 3 & -2 \\ -4 & -1 & 2 \\ 13 & 9 & -3 \end{bmatrix}$.

(a) Evaluate e^{tA}.

(b) Find the general solutions of $\dfrac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$.

(c) Solve the initial value probelm $\dfrac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$.

Solution: The given matrix A is diagonalized: $A = PDP^{-1}$ with

$$P = \begin{bmatrix} 1 & -1 & 1/2 \\ -1 & 2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Part (a): We have

$$\begin{aligned} e^{tA} &= Pe^{tD}P^{-1} \\ &= \begin{bmatrix} 1 & -1 & 1/2 \\ -1 & 2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & 0 \\ -6 & -4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5e^t - e^{2t} - 3e^{-t} & 3e^t - e^{2t} - 2e^{-t} & -e^t + e^{-t} \\ -5e^t + 2e^{2t} + 3e^{-t} & -3e^t + 2e^{2t} + 2e^{-t} & e^t - e^{-t} \\ 5e^t + e^{2t} - 6e^{-t} & 3e^t + e^{2t} - 4e^{-t} & -e^t + 2e^{-t} \end{bmatrix}. \end{aligned}$$

Part (b): The general solutions to the given system are

$$\vec{\mathbf{x}}(t) = e^{tA} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix},$$

where C_1, C_2, C_3 are free parameters.

Part (c): The solution to the initial value problem is

$$\vec{\mathbf{x}}(t) = e^{tA} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -11e^t + e^{2t} + 8e^{-t} \\ 11e^t - 2e^{2t} - 8e^{-t} \\ -11e^t - e^{2t} + 16e^{-t} \end{bmatrix}.$$

Evaluation of Matrix Exponential Using Fundamental Matrix: In the case A is not diagonalizable, one approach to obtain matrix exponential is to use Jordan forms. Here, we use another approach. We have already learned how to solve the initial value problem

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0.$$

We shall compare the solution formula with $\vec{\mathbf{x}}(t) = e^{tA}\vec{\mathbf{x}}_0$ to figure out what e^{tA} is. We know the general solutions of $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}}$ are of the following structure:

$$\vec{\mathbf{x}}(t) = C_1\vec{\mathbf{x}}_1(t) + \cdots + C_n\vec{\mathbf{x}}_n(t),$$

where $\vec{\mathbf{x}}_1(t), \cdots, \vec{\mathbf{x}}_n(t)$ are n linearly independent particular solutions. The formula can be rewritten as

$$\vec{\mathbf{x}}(t) = [\,\vec{\mathbf{x}}_1(t) \quad \cdots \quad \vec{\mathbf{x}}_n(t)\,]\,\vec{C} \qquad \text{with } \vec{C} = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}.$$

For the initial value problem, $\vec{C}$ is determined by the initial condition

$$[\,\vec{\mathbf{x}}_1(0) \quad \cdots \quad \vec{\mathbf{x}}_n(0)\,]\,\vec{C} = \vec{\mathbf{x}}_0 \quad \Longrightarrow \quad \vec{C} = [\,\vec{\mathbf{x}}_1(0) \quad \cdots \quad \vec{\mathbf{x}}_n(0)\,]^{-1}\,\vec{\mathbf{x}}_0.$$

Thus, the solution of the initial value problem is given by

$$\vec{\mathbf{x}}(t) = [\,\vec{\mathbf{x}}_1(t) \quad \cdots \quad \vec{\mathbf{x}}_n(t)\,]\,[\,\vec{\mathbf{x}}_1(0) \quad \cdots \quad \vec{\mathbf{x}}_n(0)\,]^{-1}\,\vec{\mathbf{x}}_0.$$

Comparing this with $\vec{\mathbf{x}}(t) = e^{tA}\vec{\mathbf{x}}_0$, we obtain

$$e^{tA} = [\,\vec{\mathbf{x}}_1(t) \quad \cdots \quad \vec{\mathbf{x}}_n(t)\,]\,[\,\vec{\mathbf{x}}_1(0) \quad \cdots \quad \vec{\mathbf{x}}_n(0)\,]^{-1}.$$

In this method of evaluating e^{tA}, the matrix $M(t) = [\,\vec{\mathbf{x}}_1(t) \quad \cdots \quad \vec{\mathbf{x}}_n(t)\,]$ plays an essential role. Indeed, $e^{tA} = M(t)M(0)^{-1}$.

Definition (Fundamental Matrix Solution): If $\vec{\mathbf{x}}_1(t), \cdots, \vec{\mathbf{x}}_n(t)$ are n linearly independent solutions of the n dimensional homogeneous linear system $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}}$, we call

$$M(t) = [\,\vec{\mathbf{x}}_1(t) \quad \cdots \quad \vec{\mathbf{x}}_n(t)\,]$$

a *fundamental matrix solution* of the system.

(Remark 1: The matrix function $M(t)$ satisfies the equation $M'(t) = AM(t)$. Moreover, $M(t)$ is an invertible matrix for every t. These two properties characterize fundamental matrix solutions.)

(Remark 2: Given a linear system, fundamental matrix solutions are not unique. However, when we make any choice of a fundamental matrix solution $M(t)$ and compute $M(t)M(0)^{-1}$, we always get the same result.)

EXAMPLE 2. Evaluate e^{tA} for $A = \begin{bmatrix} -7 & -9 & 9 \\ 3 & 5 & -3 \\ -3 & -3 & 5 \end{bmatrix}$.

Solution: We first solve $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}}$. We obtain

$$\vec{\mathbf{x}}(t) = C_1 e^{-t} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

This gives a fundamental matrix solution:

$$M(t) = \begin{bmatrix} 3e^{-t} & -e^{2t} & e^{2t} \\ -e^{-t} & e^{2t} & 0 \\ e^{-t} & 0 & e^{2t} \end{bmatrix}.$$

The matrix exponential is

$$\begin{aligned} e^{tA} = M(t)M(0)^{-1} &= \begin{bmatrix} 3e^{-t} & -e^{2t} & e^{2t} \\ -e^{-t} & e^{2t} & 0 \\ e^{-t} & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 3e^{-t} - 2e^{2t} & 3e^{-t} - 3e^{2t} & -3e^{-t} + 3e^{2t} \\ -e^{-t} + 2e^{2t} & -e^{-t} + 2e^{2t} & e^{-t} - e^{2t} \\ e^{-t} - e^{2t} & e^{-t} - e^{2t} & -e^{-t} + 2e^{2t} \end{bmatrix}. \end{aligned}$$

EXAMPLE 3. Evaluate e^{tA} for $A = \begin{bmatrix} -5 & -8 & 4 \\ 2 & 3 & -2 \\ 6 & 14 & -5 \end{bmatrix}$.

Solution: We first solve $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}}$. We obtain

$$\vec{\mathbf{x}}(t) = C_1 e^{-t} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{-3t} \begin{bmatrix} -1 - 2t \\ 1/2 + t \\ t \end{bmatrix}.$$

This gives a fundamental matrix solution:

$$M(t) = \begin{bmatrix} 3e^{-t} & -2e^{-3t} & -e^{-3t} - 2te^{-3t} \\ -e^{-t} & e^{-3t} & \frac{1}{2}e^{-3t} + te^{-3t} \\ e^{-t} & e^{-3t} & te^{-3t} \end{bmatrix}.$$

The matrix exponential is

$$\begin{aligned} e^{tA} = M(t)M(0)^{-1} &= \begin{bmatrix} 3e^{-t} & -2e^{-3t} & -e^{-3t} - 2te^{-3t} \\ -e^{-t} & e^{-3t} & \frac{1}{2}e^{-3t} + te^{-3t} \\ e^{-t} & e^{-3t} & te^{-3t} \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ -1 & 1 & 1/2 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 3e^{-t} - 2e^{-3t} - 8te^{-3t} & 6e^{-t} - 6e^{-3t} - 20te^{-3t} & 4te^{-3t} \\ -e^{-t} + e^{-3t} + 4te^{-3t} & -2e^{-t} + 3e^{-3t} + 10te^{-3t} & -2te^{-3t} \\ e^{-t} - e^{-3t} + 4te^{-3t} & 2e^{-t} - 2e^{-3t} + 10te^{-3t} & e^{-3t} - 2te^{-3t} \end{bmatrix}. \end{aligned}$$

EXAMPLE 4. Evaluate e^{tA} for $A = \begin{bmatrix} 9 & -5 \\ 4 & 5 \end{bmatrix}$.

Solution 1: (Use Diagonalization)

Solving $\det(A - \lambda I) = 0$, we obtain the eigenvalues of A: $\lambda_1 = 7 + 4i, \lambda_2 = 7 - 4i$.

Eigenvectors for $\lambda_1 = 7 + 4i$: are obtained by solving $[A - (7 + 4i)I]\vec{\mathbf{v}} = 0$:

$$\vec{\mathbf{v}} = v_2 \begin{bmatrix} \frac{1}{2} + i \\ 1 \end{bmatrix}. \qquad (*)$$

Eigenvectors for $\lambda_2 = 7 - 4i$: are complex conjugate of the vectors in $(*)$.

The matrix A is now diagonalized: $A = PDP^{-1}$ with

$$P = \begin{bmatrix} \frac{1}{2} + i & \frac{1}{2} - i \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 7 + 4i & 0 \\ 0 & 7 - 4i \end{bmatrix}.$$

We have

$$\begin{aligned} e^{tA} &= Pe^{tD}P^{-1} \\ &= \begin{bmatrix} \frac{1}{2} + i & \frac{1}{2} - i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(7+4i)t} & 0 \\ 0 & e^{(7-4i)t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}i & \frac{1}{2} + \frac{1}{4}i \\ \frac{1}{2}i & \frac{1}{2} - \frac{1}{4}i \end{bmatrix} \\ &= \begin{bmatrix} (\frac{1}{2} - \frac{1}{4}i)e^{(7+4i)t} + (\frac{1}{2} + \frac{1}{4}i)e^{(7-4i)t} & \frac{5}{8}ie^{(7+4i)t} - \frac{5}{8}ie^{(7-4i)t} \\ -\frac{1}{2}ie^{(7+4i)t} + \frac{1}{2}ie^{(7-4i)t} & (\frac{1}{2} + \frac{1}{4}i)e^{(7+4i)t} + (\frac{1}{2} - \frac{1}{4}i)e^{(7-4i)t} \end{bmatrix}. \end{aligned}$$

Solution 2: (Use fundamental solutions and complex exp functions)

A fundamental matrix solution can be obtained from the eigenvalues and eigenvectors:

$$M(t) = \begin{bmatrix} (\frac{1}{2} + i)e^{(7+4i)t} & (\frac{1}{2} - i)e^{(7-4i)t} \\ e^{(7+4i)t} & e^{(7-4i)t} \end{bmatrix}.$$

The matrix exponential is

$$\begin{aligned} e^{tA} &= M(t)M(0)^{-1} = \begin{bmatrix} (\frac{1}{2} + i)e^{(7+4i)t} & (\frac{1}{2} - i)e^{(7-4i)t} \\ e^{(7+4i)t} & e^{(7-4i)t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} + i & \frac{1}{2} - i \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (\frac{1}{2} - \frac{1}{4}i)e^{(7+4i)t} + (\frac{1}{2} + \frac{1}{4}i)e^{(7-4i)t} & \frac{5}{8}ie^{(7+4i)t} - \frac{5}{8}ie^{(7-4i)t} \\ -\frac{1}{2}ie^{(7+4i)t} + \frac{1}{2}ie^{(7-4i)t} & (\frac{1}{2} + \frac{1}{4}i)e^{(7+4i)t} + (\frac{1}{2} - \frac{1}{4}i)e^{(7-4i)t} \end{bmatrix}. \end{aligned}$$

Solution 3: (Use fundamental solutions and avoid complex exp functions)

A fundamental matrix solution can be obtained from the eigenvalues and eigenvectors:

$$M(t) = \begin{bmatrix} e^{7t}\left(\frac{1}{2}\cos 4t - \sin 4t\right) & e^{7t}\left(\cos 4t + \frac{1}{2}\sin 4t\right) \\ e^{7t}\cos 4t & e^{7t}\sin 4t \end{bmatrix}.$$

The matrix exponential is

$$\begin{aligned} e^{tA} &= M(t)M(0)^{-1} = \begin{bmatrix} e^{7t}\left(\frac{1}{2}\cos 4t - \sin 4t\right) & e^{7t}\left(\cos 4t + \frac{1}{2}\sin 4t\right) \\ e^{7t}\cos 4t & e^{7t}\sin 4t \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} e^{7t}\cos 4t + \frac{1}{2}e^{7t}\sin 4t & -\frac{5}{4}e^{7t}\sin 4t \\ e^{7t}\sin 4t & e^{7t}\cos 4t - \frac{1}{2}e^{7t}\sin 4t \end{bmatrix}. \end{aligned}$$

APPENDIX: Common Mistakes

Since the years of freshmen calculus, we all loved the exponential function e^x with scalar variable x. There are tons of simple and beautiful formulas for the scalar function e^x. The matrix exponential is, however, a quite different beast. We need to be a little careful in handling it.

Here are some common mistakes I have seen people make:

- $e^{t(A+B)} = e^{tA}e^{tB}$.
- The solutions of $\dfrac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}}$ are $\vec{\mathbf{x}}(t) = e^{tA(t)}\vec{\mathbf{C}}$.
- The solutions of $\dfrac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}}$ are $\vec{\mathbf{x}}(t) = e^{\int_0^t A(s)ds}\vec{\mathbf{C}}$.
- $\left(e^{B(t)}\right)' = B'(t)e^{B(t)}$.

All the above four statements are WRONG!

(1) It is true that $e^{t(A+B)} = e^{tA}e^{tB}$ if $AB = BA$. But in general, $e^{t(A+B)} \neq e^{tA}e^{tB}$.

Example: For

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A+B = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix},$$

we have

$$e^{tA}e^{tB} = \frac{1}{2}\begin{bmatrix} 1+e^{2t} & 1-e^{2t} \\ 1-e^{2t} & 1+e^{2t} \end{bmatrix}\begin{bmatrix} e^{-t} & 1-e^{-t} \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-t}+e^{t} & 2-e^{-t}-e^{t} \\ e^{-t}-e^{t} & 2-e^{-t}+e^{t} \end{bmatrix},$$

but

$$e^{t(A+B)} = \begin{bmatrix} 1 & 0 \\ 1-e^{t} & e^{t} \end{bmatrix}.$$

(2) We learned how to solve

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} \qquad \text{where } A \text{ is a constant matrix.}$$

Unfortunately there is no general solution method for

$$(*) \qquad \frac{d\vec{\mathbf{x}}}{dt} = A(t)\vec{\mathbf{x}} \qquad \text{where } A(t) \text{ is nonconstant.}$$

In particular,

$$\vec{\mathbf{x}}(t) = e^{tA(t)}\vec{\mathbf{C}}.$$

does not solve $(*)$. This even fails in the scalar case.

Example: The solutions of the scalar equation $\dfrac{dx}{dt} = (\sin t)x$ is given by $x(t) = e^{-\cos t}C$, but not $e^{t\sin t}C$.

(3) Likewise,

$$\vec{\mathbf{x}}(t) = e^{\int_0^t A(s)ds}\vec{\mathbf{C}}$$

is also a *wrong* solution formula for $(*)$. This formula is only valid for scalar equations, i.e., when the space dimension is 1.

Example: Consider

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & 0 \\ 2t & -1 \end{bmatrix}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{where } A(t) = \begin{bmatrix} 1 & 0 \\ 2t & -1 \end{bmatrix}.$$

The solution of this initial value problem is

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} e^t \\ (t-\frac{1}{2})e^t + \frac{1}{2}e^{-t} \end{bmatrix}.$$

The function

$$\exp\left(\int_0^t A(s)ds\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

does not give the solution. Indeed,

$$\begin{aligned} \exp\left(\int_0^t A(s)ds\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \exp\left(\begin{bmatrix} t & 0 \\ t^2 & -t \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} e^t & 0 \\ \frac{1}{2}te^t - \frac{1}{2}te^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ \frac{1}{2}te^t - \frac{1}{2}te^{-t} \end{bmatrix}. \end{aligned}$$

(4) We do have:

$(e^{b(t)})' = b'(t)e^{b(t)}$ holds for scalar functions $b(t)$, and
$(e^{tA})' = Ae^{tA} = e^{tA}A$ for constant matrices A.

But, in general, for nonconstant matrix function $B(t)$,

$(e^{B(t)})'$ is neither $B'(t)e^{B(t)}$ nor $e^{B(t)}B'(t)$.

Example: Let $B(t) = \begin{bmatrix} t & 0 \\ t^2 & -t \end{bmatrix}$. We have $e^{B(t)} = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}te^t - \frac{1}{2}te^{-t} & e^{-t} \end{bmatrix}$, and hence

$$\begin{aligned} \left(e^{B(t)}\right)' &= \begin{bmatrix} e^t & 0 \\ \frac{t+1}{2}e^t + \frac{t-1}{2}e^{-t} & -e^{-t} \end{bmatrix}, \\ B'(t)e^{B(t)} &= \begin{bmatrix} 1 & 0 \\ 2t & -1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ \frac{1}{2}te^t - \frac{1}{2}te^{-t} & e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ \frac{3}{2}te^t + \frac{1}{2}te^{-t} & -e^{-t} \end{bmatrix}, \\ e^{B(t)}B'(t) &= \begin{bmatrix} e^t & 0 \\ \frac{1}{2}te^t - \frac{1}{2}te^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2t & -1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}te^t + \frac{3}{2}te^{-t} & -e^{-t} \end{bmatrix}. \end{aligned}$$

All three matrix functions $(e^{B(t)})'$, $B'(t)e^{B(t)}$ and $e^{B(t)}B'(t)$ are different from each other.

EXERCISES

[1] Evaluate e^{tA} for $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}$.

[2] (a) Evaluate e^{tA} for $A = \begin{bmatrix} -4 & 12 \\ -3 & 8 \end{bmatrix}$.

(b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

[3] Evaluate e^{tA} for $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$.

[4] Evaluate e^{tA} for $A = \begin{bmatrix} 5 & 4 & -2 \\ -12 & -9 & 4 \\ -12 & -8 & 3 \end{bmatrix}$.

[5] (a) Evaluate e^{tA} for $A = \begin{bmatrix} 9 & 7 & -3 \\ -16 & -12 & 5 \\ -8 & -5 & 2 \end{bmatrix}$.

(b) Solve $\dfrac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

[6] Evaluate e^{tA} for $A = \begin{bmatrix} 5 & -4 \\ 2 & 1 \end{bmatrix}$.

[7] Evaluate e^{tA} for $A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

See next page for answers

Answers:

[1] $$e^{tA} = \begin{bmatrix} \frac{3}{2}e^{4t} - \frac{1}{2}e^{2t} & -\frac{3}{2}e^{4t} + \frac{3}{2}e^{2t} \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} & -\frac{1}{2}e^{4t} + \frac{3}{2}e^{2t} \end{bmatrix}$$

[2] (a) $$e^{tA} = \begin{bmatrix} e^{2t} - 6te^{2t} & 12te^{2t} \\ -3te^{2t} & e^{2t} + 6te^{2t} \end{bmatrix}$$

(b) $$\vec{\mathbf{x}}(t) = e^{tA}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 5e^{2t} - 42te^{2t} \\ -e^{2t} - 21te^{2t} \end{bmatrix}$$

[3] $$e^{tA} = \begin{bmatrix} 3e^{t} - 2e^{2t} & -5e^{t} + 6e^{2t} - e^{3t} & 3e^{t} - 4e^{2t} + e^{3t} \\ 3e^{t} - 3e^{2t} & -5e^{t} + 9e^{2t} - 3e^{3t} & 3e^{t} - 6e^{2t} + 3e^{3t} \\ 3e^{t} - 3e^{2t} & -5e^{t} + 9e^{2t} - 4e^{3t} & 3e^{t} - 6e^{2t} + 4e^{3t} \end{bmatrix}$$

[4] $$e^{tA} = \begin{bmatrix} 3e^{t} - 2e^{-t} & 2e^{t} - 2e^{-t} & -e^{t} + e^{-t} \\ -6e^{t} + 6e^{-t} & -4e^{t} + 5e^{-t} & 2e^{t} - 2e^{-t} \\ -6e^{t} + 6e^{-t} & -4e^{t} + 4e^{-t} & 2e^{t} - e^{-t} \end{bmatrix}$$

[5] (a) $$e^{tA} = \begin{bmatrix} 3e^{t} - 2e^{-t} + 4te^{-t} & 2e^{t} - 2e^{-t} + 3te^{-t} & -e^{t} + e^{-t} - te^{-t} \\ -6e^{t} + 6e^{-t} - 4te^{-t} & -4e^{t} + 5e^{-t} - 3te^{-t} & 2e^{t} - 2e^{-t} + te^{-t} \\ -6e^{t} + 6e^{-t} + 4te^{-t} & -4e^{t} + 4e^{-t} + 3te^{-t} & 2e^{t} - e^{-t} - te^{-t} \end{bmatrix}$$

(b) $$\vec{\mathbf{x}}(t) = e^{tA}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4e^{t} - 3e^{-t} + 6te^{-t} \\ -8e^{t} + 9e^{-t} - 6te^{-t} \\ -8e^{t} + 9e^{-t} + 6te^{-t} \end{bmatrix}$$

[6] $$e^{tA} = e^{3t}\begin{bmatrix} \cos 2t + \sin 2t & -2\sin 2t \\ \sin 2t & \cos 2t - \sin 2t \end{bmatrix},$$

or, equivalently,

$$e^{tA} = \frac{1}{2}\begin{bmatrix} (1-i)e^{(3+2i)t} + (1+i)e^{(3-2i)t} & 2ie^{(3+2i)t} - 2ie^{(3-2i)t} \\ -ie^{(3+2i)t} + ie^{(3-2i)t} & (1+i)e^{(3+2i)t} + (1-i)e^{(3-2i)t} \end{bmatrix}$$

[7] $$e^{tA} = \frac{1}{5}\begin{bmatrix} 2e^{-3t} + 3\cos t + \sin t & -2e^{-3t} + 2\cos t - \sin t & e^{-3t} - \cos t + 3\sin t \\ -4e^{-3t} + 4\cos t - 2\sin t & 4e^{-3t} + \cos t - 3\sin t & -2e^{-3t} + 2\cos t + 4\sin t \\ -2e^{-3t} + 2\cos t - 6\sin t & 2e^{-3t} - 2\cos t - 4\sin t & -e^{-3t} + 6\cos t + 2\sin t \end{bmatrix}$$