Diff Eqs with

- Discontinuous Terms
- Piecewise Terms
- Impulsive Terms

Example Solve $\begin{cases} y''(t) + y(t) = f(t) \\ y(0)=3,\ y'(0)=8 \end{cases}$ where $f(t) = \begin{cases} 15\sin(5t) & t < 9\pi \\ 15 & t \geq 9\pi \end{cases}$

Solution $f(t) = [1 - u(t-9\pi)]\, 15\sin(5t) + u(t-9\pi)\, 15$

$$= 15\sin(5t) - u(t-9\pi)\, 15\sin[5(t-9\pi)+45\pi] + u(t-9\pi)\, 15$$

$$= 15\sin(5t) + u(t-9\pi)\, 15\sin 5(t-9\pi) + u(t-9\pi)\, 15$$

$$\xrightarrow{\mathcal{L}} \quad s^2 Y(s) - 3s - 8 + Y(s) = \frac{15\cdot 5}{s^2+25} + e^{-9\pi s}\frac{15\cdot 5}{s^2+25} + e^{-9\pi s}\frac{15}{s},$$

$$Y(s) = \frac{3s+8}{s^2+1} + \frac{75}{(s^2+1)(s^2+25)} + e^{-9\pi s}\frac{75}{(s^2+1)(s^2+25)} + e^{-9\pi s}\frac{15}{(s^2+1)s}$$

$$= \frac{3s+8}{s^2+1} + \frac{25/8}{s^2+1} - \frac{25/8}{s^2+25} + e^{-9\pi s}\frac{25/8}{s^2+1} - e^{-9\pi s}\frac{25/8}{s^2+25} + e^{-9\pi s}\frac{15}{s} - e^{-9\pi s}\frac{15s}{s^2+1}.$$

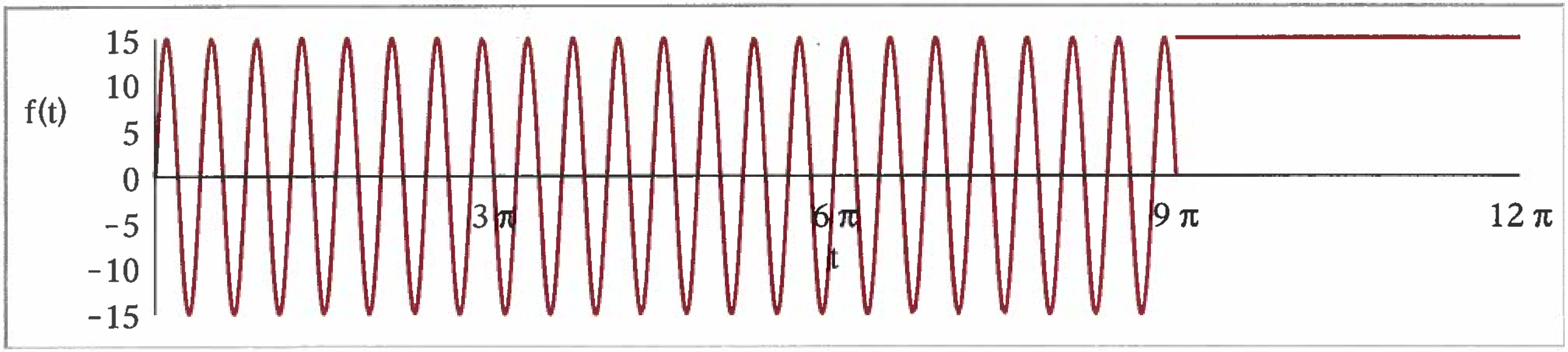
$$\xrightarrow{\mathcal{L}^{-1}}$$

$$y(t) = 3\cos t + 8\sin t + \frac{25}{8}\sin t - \frac{5}{8}\sin(5t)$$
$$+ u(t-9\pi)\left[\frac{25}{8}\sin(t-9\pi) - \frac{5}{8}\sin 5(t-9\pi) + 15 - 15\cos(t-9\pi)\right]$$

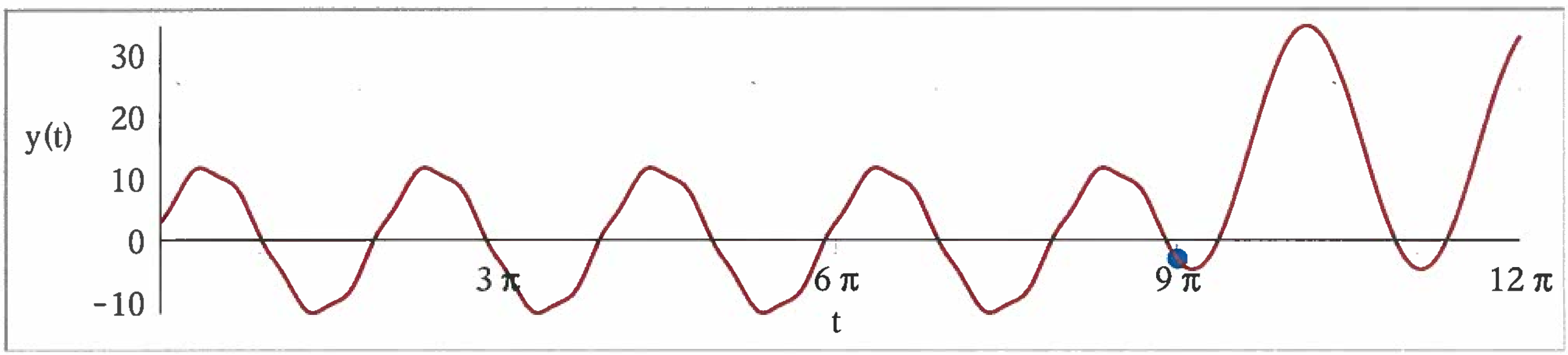
$$= 3\cos t + \frac{89}{8}\sin t - \frac{5}{8}\sin(5t) + u(t-9\pi)\left[-\frac{25}{8}\sin t + \frac{5}{8}\sin(5t) + 15 + 15\cos t\right]$$

$$= \begin{cases} 3\cos t + \frac{89}{8}\sin t - \frac{5}{8}\sin(5t) & t < 9\pi, \\ 18\cos t + 8\sin t + 15 & t \geq 9\pi. \end{cases}$$

$$f(t)=\begin{cases} 15\sin(5t) & t<9\pi \\ 15 & t\geq 9\pi \end{cases}$$



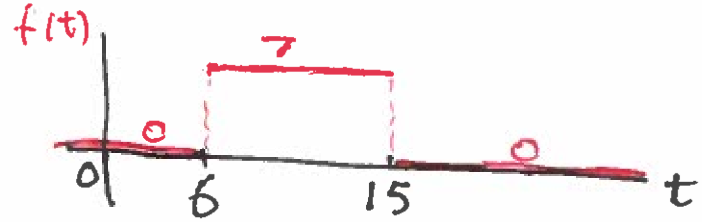
$$y(t)=\begin{cases} 3\cos t+\frac{89}{8}\sin t-\frac{5}{8}\sin(5t) & t<9\pi \\ 18\cos t+8\sin t+15 & t\geq 9\pi \end{cases}$$



Example

$$\begin{cases} y''(t) + 4y'(t) + 3y(t) = f(t), \\ y(0) = -2, \quad y'(0) = 12 \end{cases}$$

where $f(t) = \begin{cases} 0 & t < 6 \\ 7 & 6 \le t < 15 \\ 0 & t \ge 15 \end{cases}$



Solution

- $f(t) = [u(t-6) - u(t-15)]\, 7$
  $= u(t-6)\,7 - u(t-15)\,7,$

- Take the Laplace transform of the Diff Eq:

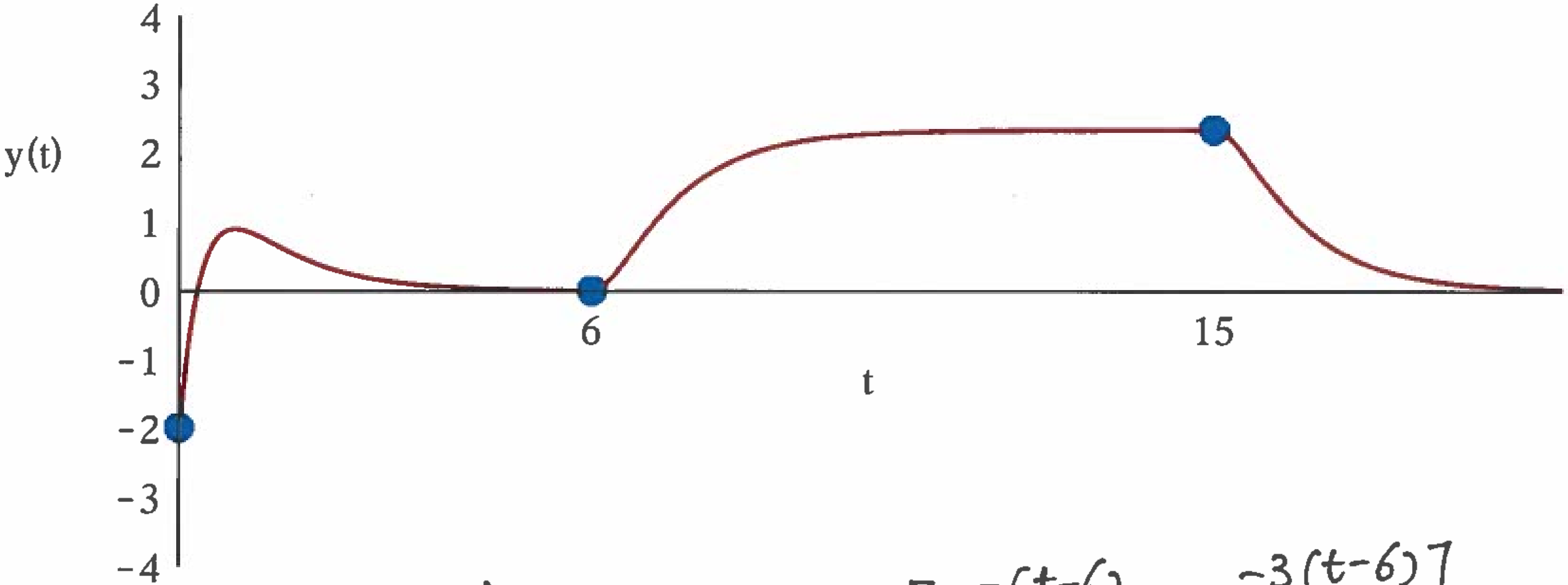
$$[s^2 Y(s) - s(-2) - 12] + 4[sY(s) - (-2)] + 3Y(s) = e^{-6s}\frac{7}{s} - e^{-15s}\frac{7}{s},$$

- $$Y(s) = \frac{-2s+4}{s^2+4s+3} + e^{-6s}\frac{7}{s(s^2+4s+3)} - e^{-15s}\frac{7}{s(s^2+4s+3)}.$$

$$= \left(\frac{3}{s+1} + \frac{-5}{s+3}\right) + e^{-6s}\left(\frac{7/3}{s} + \frac{-7/2}{s+1} + \frac{7/6}{s+3}\right) - e^{-15s}(\text{-----})$$

$s^2+4s+3 = (s+1)(s+3)$

- $$\xrightarrow{\mathcal{L}^{-1}} y(t) = (3e^{-t} - 5e^{-3t}) + u(t-6)\left[\frac{7}{3} - \frac{7}{2}e^{-(t-6)} + \frac{7}{6}e^{-3(t-6)}\right] - u(t-15)\left[\frac{7}{3} - \frac{7}{2}e^{-(t-15)} + \frac{7}{6}e^{-3(t-15)}\right].$$

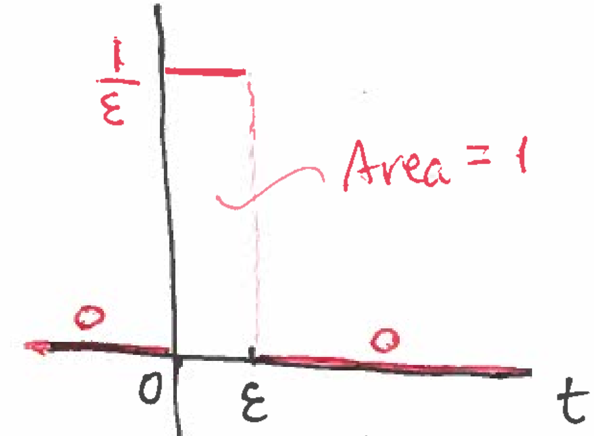


$$y(t) = 3e^{-t} - 5e^{-3t} + u(t-6)\left[\frac{7}{3} - \frac{7}{2}e^{-(t-6)} + \frac{7}{6}e^{-3(t-6)}\right] - u(t-15)\left[\frac{7}{3} - \frac{7}{2}e^{-(t-15)} + \frac{7}{6}e^{-3(t-15)}\right]$$

$$= \begin{cases} 3e^{-t} - 5e^{-3t}, & t < 6 \\ 3e^{-t} - 5e^{-3t} + \frac{7}{3} - \frac{7}{2}e^{-(t-6)} + \frac{7}{6}e^{-3(t-6)}, & 6 \le t < 15 \\ 3e^{-t} - 5e^{-3t} - \frac{7}{2}e^{-(t-6)} + \frac{7}{6}e^{-3(t-6)} + \frac{7}{2}e^{-(t-15)} - \frac{7}{6}e^{-3(t-15)}, & t \ge 15 \end{cases}$$

<u>Example</u>

$$\begin{cases} y_\varepsilon''(t) + 4y_\varepsilon(t) = [u(t)-u(t-\varepsilon)]\dfrac{1}{\varepsilon} \\ y_\varepsilon(0)=0, \quad y_\varepsilon'(0)=0 \end{cases}$$



Nearly Impulse Force

<u>Solution</u>

$\downarrow \mathcal{L}$

$$s^2 Y_\varepsilon(s) + 4Y_\varepsilon(s) = \frac{1}{\varepsilon s} - e^{-\varepsilon s}\frac{1}{\varepsilon s},$$

$$Y_\varepsilon(s) = \frac{1}{\varepsilon s(s^2+4)} - e^{-\varepsilon s}\frac{1}{\varepsilon s(s^2+4)}$$

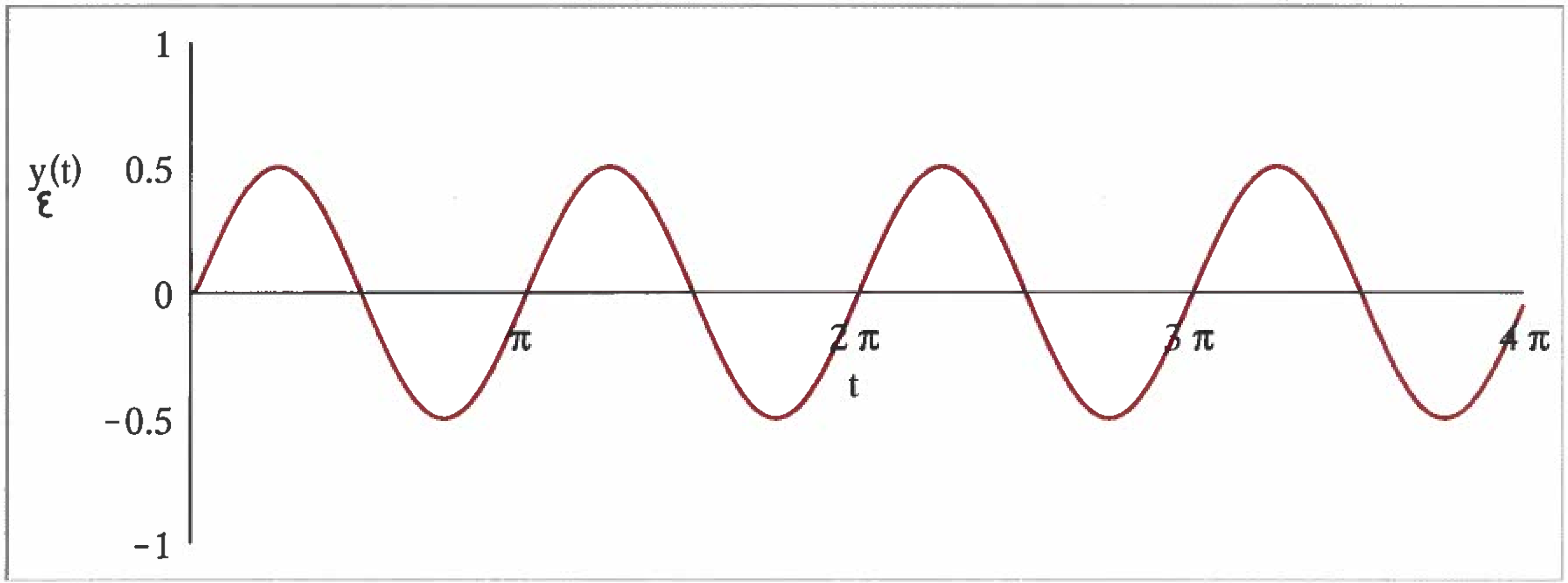
⇐ (partial fractions)

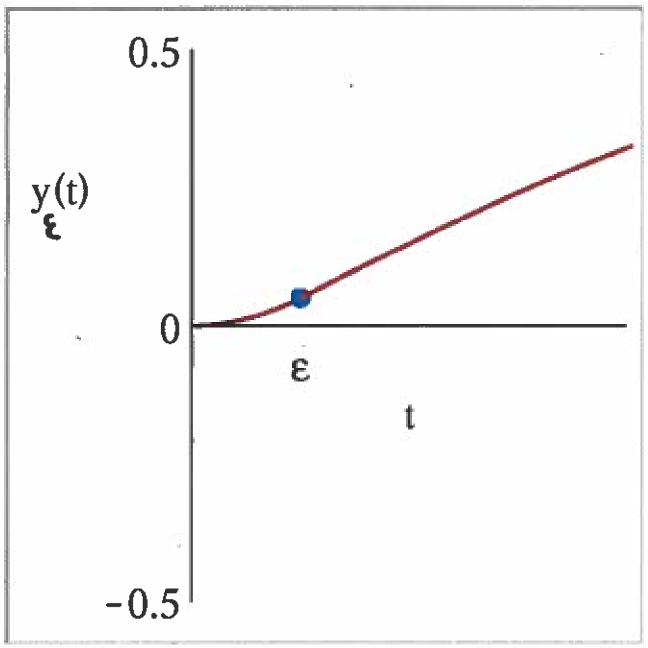
$$= \frac{1}{\varepsilon}\left(\frac{1/4}{s} - \frac{s/4}{s^2+4}\right) - \frac{e^{-\varepsilon s}}{\varepsilon}\left(\frac{1/4}{s} - \frac{s/4}{s^2+4}\right).$$

$\downarrow \mathcal{L}^{-1}$

$$y_\varepsilon(t) = \frac{1}{\varepsilon}\left[\frac{1}{4} - \frac{1}{4}\cos(2t)\right] - \frac{1}{\varepsilon}u(t-\varepsilon)\left[\frac{1}{4} - \frac{1}{4}\cos 2(t-\varepsilon)\right]$$

$$= \begin{cases} \dfrac{1}{\varepsilon}\left[\dfrac{1}{4} - \dfrac{1}{4}\cos 2t\right] & 0 \le t < \varepsilon \\ \dfrac{1}{\varepsilon}\left[\dfrac{1}{4}\cos 2(t-\varepsilon) - \dfrac{1}{4}\cos 2t\right] & t \ge \varepsilon \end{cases}$$





$$y_\varepsilon(t) = \frac{1-\cos 2t}{4\varepsilon} - u(t-\varepsilon)\,\frac{1-\cos 2(t-\varepsilon)}{4\varepsilon}$$

$$= \begin{cases} \dfrac{1-\cos 2t}{4\varepsilon} & 0 \le t < \varepsilon \\[2ex] \dfrac{\cos 2(t-\varepsilon) - \cos 2t}{4\varepsilon} & t \geq \varepsilon \end{cases}$$

For $t \geq \varepsilon$ (i.e. for a large portion of t)

$$y_\varepsilon(t) \approx \underbrace{\frac{1}{2}\sin 2t}$$

Impulse Response