Convolution $(f * g)(t)$

Definition Given $f(t)$ & $g(t)$ on $0 \le t < \infty$.

the **convolution** of f & g is :

$$(f * g)(t) = \int_{\tau=0}^{\tau=t} f(t-\tau)\, g(\tau)\, d\tau .$$

Theorem If $f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)$, $g(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} G(s)$

then

$$(f * g)(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)\, G(s) .$$

Example. Let $f(t) = e^{3t}$, $g(t) = e^{7t}$.

(a) Find $(f*g)(t)$, using the definition of convolution.

(b) Find $(g*f)(t)$, ...

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$$= \left[\frac{1}{4}e^{3t}e^{4\tau}\right]_{\tau=0}^{\tau=t} = \boxed{\frac{1}{4}e^{7t} - \frac{1}{4}e^{3t}}$$

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General Properties of Convolution

$$f*0 = 0*f = 0,$$

$$f*g = g*f,$$

$$(f*g)*h = f*(g*h)$$

$$f*(c_1 g_1 + c_2 g_2) = c_1 f*g_1 + c_2 f*g_2$$

Warning

- $1*f \neq f$, in general
- $\delta(t)*f(t) = f(t)$
- $(f*g)(t) \neq f(t)\,g(t)$, in general