Stability in n-dim Homog. Lin. Systems with Constant coefficients

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \text{where } \vec{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A: n \times n \text{ Const. matrix}$$

- $\vec{x} = \vec{0}$ is an equilibrium.
- Eigenvalue λ with $\mathrm{Re}\,\lambda < 0 \Rightarrow$ Helps stabilization
- Eigenvalue λ with $\mathrm{Re}\,\lambda > 0 \Rightarrow$ Implies instability
- Eigenvalue λ with $\mathrm{Re}\,\lambda = 0$: Neutral Eigenvalues

① If all eigenvalues λ have $\mathrm{Re}\,\lambda < 0$,
then $\vec{x} = \vec{0}$ is asymptotically stable

② If one of the eigenvalues λ has $\mathrm{Re}\,\lambda > 0$,
then $\vec{x} = \vec{0}$ is unstable.

③ If all eigenvalues λ have $\mathrm{Re}\,\lambda \leq 0$, and
any eigenvalue λ with $\mathrm{Re}\,\lambda = 0$ is non-defective,
then $\vec{x} = \vec{0}$ is stable, but not asymptotically stable.

④ If all eigenvalues λ have $\mathrm{Re}\,\lambda \leq 0$, and
one of the eigenvalues λ with $\mathrm{Re}\,\lambda = 0$ is defective,
then $\vec{x} = \vec{0}$ is unstable.

$\boxed{\text{The System under Consideration: } \dfrac{d\vec{x}}{dt} = A\vec{x}}$

List of Examples

- All eigenvalues have $\mathrm{Re}\,\lambda < 0$: Examples 1, 6, 8, 10.

- One of the eigenvalues has $\mathrm{Re}\,\lambda > 0$: Examples 2, 4, 5, 12.

- [All eigenvalues have $\mathrm{Re}\,\lambda \le 0$ & all neutral eigenvalues are non-defective] : Examples 3, 7, 11.

- [All eigenvalues have $\mathrm{Re}\,\lambda \le 0$ & a neutral eigenvalue is defective] : Examples 9, 13.

- [All eigenvalues have $\mathrm{Re}\,\lambda \le 0$ & a neutral eigenvalue is repeated] : Examples 7, 9, 13.

- Some eigenvalues are complex eigenvalues : Examples 10, 11, 12.

- A five dimensional example : Example 13.

Example 1 $\dfrac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 3 & 3 & -2 \\ -4 & -4 & 2 \\ 13 & 9 & -6 \end{bmatrix}$

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues of A: $\det(A - \lambda I) = 0$, ..., solve, ...

$\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -4$

· All eigenvalues < 0

$\Rightarrow \vec{x} = \vec{0}$ is asymptotically stable.

Example 1

$$A = \begin{bmatrix} 3 & 3 & -2 \\ -4 & -4 & 2 \\ 13 & 9 & -6 \end{bmatrix}$$

$$\vec{x}(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + C_3 e^{-4t} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Example 2 $\frac{d\vec{x}}{dt} = A\vec{x}$, $A = \begin{bmatrix} 20 & 14 & -5 \\ -23 & -17 & 5 \\ 26 & 16 & -9 \end{bmatrix}$

Is $\vec{x} = \vec{0}$ stable, asymp. stable, or unstable?

Solution Eigenvalues of A:

$\lambda_1 = 1$, $\lambda_2 = -3$, $\lambda_3 = -4$.

· Since $\lambda_1 = 1 > 0$,

$\vec{x} = \vec{0}$ is unstable.

Example 2

$$A = \begin{bmatrix} 20 & 14 & -5 \\ -23 & -17 & 5 \\ 26 & 16 & -9 \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-4t} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Example 3 $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 4 & 3 & -2 \\ -4 & -3 & 2 \\ 13 & 9 & -5 \end{bmatrix}$

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues of A:

$\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -3$

- All eigenvalues are real and ≤ 0.
- Neutral eigenvalue $\lambda_1 = 0$ is simple and hence non-defective.

$\Rightarrow$ $\vec{x} = \vec{0}$ is stable, but not asymptotically stable.

Example 3

$$A = \begin{bmatrix} 4 & 3 & -2 \\ -4 & -3 & 2 \\ 13 & 9 & -5 \end{bmatrix}$$

$$\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

5. Solve the differential equation:

$$y' + 4xy = 4x\sqrt{y},$$

and give an *explicit* solution.

Example 4 $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 11 & 7 & -3 \\ -11 & -7 & 3 \\ 17 & 11 & -5 \end{bmatrix}$.

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues of A:

$\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = -2$.

Since $\lambda_1 > 0$,

$\vec{x} = \vec{0}$ is unstable.

Example 4

$$A = \begin{bmatrix} 11 & 7 & -3 \\ -11 & -7 & 3 \\ 17 & 11 & -5 \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Example 5 $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} -7 & -9 & 9 \\ 3 & 5 & -3 \\ -3 & -3 & 5 \end{bmatrix}$.

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues of A: $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 2$

Since $\lambda_2 = \lambda_3 > 0$, $\vec{x} = \vec{0}$ is unstable

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Example 6 $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} -12 & -9 & 9 \\ 3 & 0 & -3 \\ -3 & -3 & 0 \end{bmatrix}$.

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues of A: $\lambda_1 = -6$, $\lambda_2 = \lambda_3 = -3$

Since all eigenvalues < 0,
$\vec{x} = \vec{0}$ is asymptotically stable.

Example 7 $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} -9 & -9 & 9 \\ 3 & 3 & -3 \\ -3 & -3 & 3 \end{bmatrix}$.

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues of A: $\lambda_1 = -3$, $\lambda_2 = \lambda_3 = 0$.

- All eigenvalues ≤ 0.
- Neutral eigenvalue $\lambda_2 = \lambda_3 = 0$:
  - Multiplicity $= 2$
  - dim of the eigen space: Solve $A\vec{x} = 0$, ..., $\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(dim of the eigenspace) $= 2$

Thus, $\lambda_2 = \lambda_3 = 0$ is Non-defective.

$\Rightarrow$ $\vec{x} = \vec{0}$ is stable, but not asymptotically stable.

Example 5

$$A = \begin{bmatrix} -7 & -9 & 9 \\ 3 & 5 & -3 \\ -3 & -3 & 5 \end{bmatrix} \qquad \vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Example 6

$$A = \begin{bmatrix} -12 & -9 & 9 \\ 3 & 0 & -3 \\ -3 & -3 & 0 \end{bmatrix} \qquad \vec{x}(t) = c_1 e^{-6t}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Example 7

$$A = \begin{bmatrix} -9 & -9 & 9 \\ 3 & 3 & -3 \\ -3 & -3 & 3 \end{bmatrix} \qquad \vec{x}(t) = c_1 e^{-3t}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Example 8 $\frac{d\vec{x}}{dt} = A\vec{x}$, $A = \begin{bmatrix} -5 & -8 & 4 \\ 2 & 3 & -2 \\ 6 & 14 & -5 \end{bmatrix}$

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues of A: $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = -3$

- All eigenvalues $< 0 \Rightarrow \vec{x} = \vec{0}$ is asymptotically stable.

Example 9 $\frac{d\vec{x}}{dt} = A\vec{x}$, $A = \begin{bmatrix} 2 & 8 & -4 \\ -2 & -6 & 2 \\ -6 & -14 & 2 \end{bmatrix}$

Is $\vec{x} = \vec{0}$ stable, asymp. stable, or unstable?

Solution Eigenvalues of A: $\lambda_1 = -2$, $\lambda_2 = \lambda_3 = 0$.

- All eigenvalues ≤ 0
- Neutral Eigenvalues: Defective or Non-Defective

$$\left.\begin{array}{l} \lambda_2 = \lambda_3 = 0 : \text{Multiplicity} = 2 \\ \text{Solve } A\vec{x} = 0 \Rightarrow \vec{x} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{l}\text{Dim. of the Eigenspace} \\ \text{for Neutral Eigenvalue}\end{array}\right] = 1 \end{array}\right\} \begin{array}{l}\lambda_2 = \lambda_3 = 0 \\ \text{is Defective}\end{array}$$

- Thus, $\vec{x} = \vec{0}$ is unstable.

Example 8

$$A = \begin{bmatrix} -5 & -8 & 4 \\ 2 & 3 & -2 \\ 6 & 14 & -5 \end{bmatrix} \qquad \vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -2-4t \\ 1+2t \\ 2t \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 4t \\ -2t \\ 1-2t \end{bmatrix}$$

Example 9

$$A = \begin{bmatrix} 2 & 8 & -4 \\ -2 & -6 & 2 \\ -6 & -14 & 2 \end{bmatrix} \qquad \vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2+4t \\ 1-2t \\ -2t \end{bmatrix} + c_3 \begin{bmatrix} -4t \\ 2t \\ 1+2t \end{bmatrix}$$

Example 10 $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 3 & 22 & -36 \\ 2 & 5 & -18 \\ 2 & 7 & -17 \end{bmatrix}$

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues of A: $\lambda_1 = -5$, $\lambda_2 = -2+3i$, $\lambda_3 = -2-3i$

All eigenvalues have $\operatorname{Re}\lambda < 0$ $\Rightarrow$ $\vec{x} = \vec{0}$ is asymp. stable.

Example 11 $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 5 & 22 & -36 \\ 2 & 7 & -18 \\ 2 & 7 & -15 \end{bmatrix}$

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues of A: $\lambda_1 = -3$, $\lambda_2 = 3i$, $\lambda_3 = -3i$

- All eigenvalues have $\operatorname{Re}\lambda \leq 0$
- Neutral eigenvalues $\lambda_2 = 3i$, $\lambda_3 = -3i$ are simple & hence non-defective

$\Rightarrow$ $\vec{x} = \vec{0}$ is stable, but not asymp. stable.

Example 12 $\dfrac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 6 & 22 & -36 \\ 2 & 8 & -18 \\ 2 & 7 & -14 \end{bmatrix}$

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

~~Solution~~ Eigenvalues of A:

$\lambda_1 = -2$, $\lambda_2 = 1+3i$, $\lambda_3 = 1-3i$

- $\operatorname{Re} \lambda_2 = 1 > 0$

$\Rightarrow$ $\vec{x} = \vec{0}$ is unstable.

Example 10

$$A = \begin{bmatrix} 3 & 22 & -36 \\ 2 & 5 & -18 \\ 2 & 7 & -17 \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{-5t}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{-2t}\left\{ \cos 3t \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\} + c_3 e^{-2t}\left\{ \sin 3t \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \cos 3t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Example 11

$$A = \begin{bmatrix} 5 & 22 & -36 \\ 2 & 7 & -18 \\ 2 & 7 & -15 \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{-3t}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_2 \left\{ \cos 3t \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\} + c_3 \left\{ \sin 3t \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \cos 3t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Example 12

$$A = \begin{bmatrix} 6 & 22 & -36 \\ 2 & 8 & -18 \\ 2 & 7 & -14 \end{bmatrix}.$$

$$\vec{x}(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{t}\left\{ \cos 3t \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\} + c_3 e^{t}\left\{ \sin 3t \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \cos 3t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Example 13 $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 2 & -3 & -4 & 3 & -2 \\ 3 & -2 & -3 & 2 & 0 \\ 2 & -3 & -4 & 3 & -2 \\ -1 & 0 & 1 & 0 & 0 \\ -4 & 2 & 4 & -2 & 0 \end{bmatrix}$.

Is $\vec{x} = \vec{0}$ stable, asymptotically stable, or unstable?

Solution Eigenvalues: $\lambda_1 = \lambda_2 = -2$. $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

- All eigenvalues ≤ 0.
- Neutral eigenvalues $\lambda_3 = \lambda_4 = \lambda_5 = 0$: Defective or Non-Defective?

$$\begin{cases} \cdot\ A\vec{x} = \vec{0} \xrightarrow{\text{Solve}} \vec{x} = x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow (\text{Dim of the Eigenspace}) = 2. \\ \cdot\ (\text{Multiplicity of the Eigenvalue}) = 3 \end{cases}$$

$\lambda_3 = \lambda_4 = \lambda_5 = 0$ is Defective

$\Rightarrow \vec{x} = \vec{0}$ is unstable.

Example 13

$$\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} t \\ -1 \\ t \\ 0 \\ 1 \end{bmatrix}$$

$$+ c_3 \begin{bmatrix} \frac{1}{2} \\ -1 + \frac{1}{2}t \\ 1 \\ \frac{1}{2}t \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_5 \begin{bmatrix} -\frac{1}{2} \\ -1 + \frac{1}{2}t \\ 0 \\ \frac{1}{2}t \\ 1 \end{bmatrix}$$