Nonlinear System of Diff Eq's

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$$

with equilibrium $\vec{x} = \vec{a}$

Lin. Approx. System near $\vec{x} \approx \vec{a}$

$$\frac{d\vec{x}}{dt} = \boxed{}\,(\vec{x} - \vec{a})$$

where $\boxed{} = [\text{Jacobian}]\Big|_{\substack{\text{evaluated} \\ \text{at } \vec{x} = \vec{a}}}$

- The two local dynamics near $\vec{x} \approx \vec{a}$ are equivalent, provided that $\boxed{}$ has no neutral eigenvalues
  (i.e. All eigenvalues of $\boxed{}$ have $\mathrm{Re}\,\lambda \neq 0$)

- In case $\boxed{}$ has neutral eigenvalues, (i.e. $\mathrm{Re}\,\lambda = 0$ for some λ)
  the two local dynamics near $\vec{x} \approx \vec{a}$
  may be equivalent / may be non-equivalent.
    - In most cases with neutral eigenvalues, they are Non-Equivalent.
    - Need to use other methods to determine

Nonlinear Diff Eq.

$$\frac{dy}{dt} = f(y)$$

with equilibrium $y=b$

Lin. Approx. Diff Eq Near $y \approx b$

$$\frac{dy}{dt} = \boxed{}\,(y-b)$$

where $\boxed{} = \left.\frac{df}{dy}\right|_{\text{evaluated at } y=b}$

- The two local dynamics near $y \approx b$ are equivalent, provided that $\boxed{} \neq 0$.

- In case $\boxed{} = 0$, the two local dynamics near $y \approx b$ may be equivalent / may be non-equivalent.
  - In most cases of $\boxed{} = 0$, they are non-equivalent
  - Need to use other methods to determine

Example Nonlinear Diff Eq $\frac{dy}{dt} = -y^3 + y^2 + 2y$. $f(y) = -y^3 + y^2 + 2y$

• **Equilibria:** $-y^3 + y^2 + 2y = 0$, $-y(y^2 - y - 2) = -y(y-2)(y+1)$

$y = 0$, $y = 2$, $y = -1$.

• **Near $y \approx 0$:** $\frac{df}{dy} = -3y^2 + 2y + 2$, $\left.\frac{df}{dy}\right|_{y=0} = 2$.

Lin. Approx. Diff Eq. Near $y \approx 0$: $\frac{dy}{dt} = 2y$

In this Lin. Approx. Eq. $y = 0$ is repulsive & unstable

Hence, in the nonlinear diff eq $\frac{dy}{dt} = f(y)$,
$y = 0$ is repulsive & unstable

• **Near $y \approx 2$:** $\left.\frac{df}{dy}\right|_{y=2} = -6$

Lin. Approx. Diff Eq Near $y \approx 2$: $\frac{dy}{dt} = -6(y-2)$

In this Lin. Approx Eq, $y = 2$ is attractive & asymp. stable

Hence, in the nonlin. diff eq $\frac{dy}{dt} = f(y)$,
$y = 2$ is attractive & asymp. stable.

• Near $y \approx -1$: $\left.\frac{df}{dy}\right|_{y=-1} = \left.(-3y^2+2y+2)\right|_{y=-1} = -3$

Lin. Approx. Diff Eq. Near $y \approx -1$: $\frac{dy}{dt} = -3(y+1)$

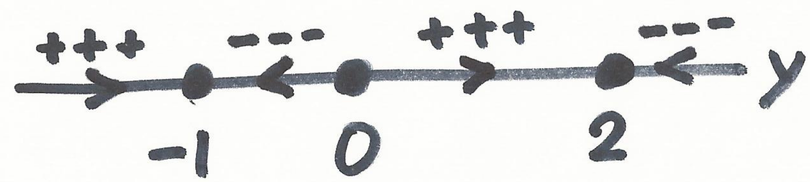
In this Lin. Approx Eq., $y=-1$ is attractive & asymp. stable.

In the Nonlin. Diff Eq $\frac{dy}{dt} = -y^3+y^2+2y$.

$y=-1$ is attractive & asymp. stable.

• We can use the sign-changing method to verify the above:

sign of $f(y) = -y^3+y^2+2y$:



Example Nonlinear Diff Eq $\frac{dy}{dt} = (y-1)^3$ $f(y) = (y-1)^3$.

· Equilibria: $(y-1)^3 = 0 \Rightarrow y = 1$.

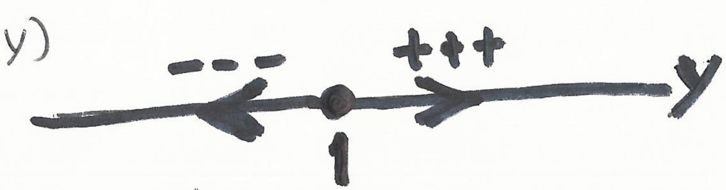
· Near $y \approx 1$: $\frac{df}{dy} = 3(y-1)^2$. $\left.\frac{df}{dy}\right|_{y=1} = 0$.

Lin. Approx. Diff Eq. Near $y \approx 1$: $\frac{dy}{dt} = 0(y-1)$. i.e. $\frac{dy}{dt} = 0$.

In this Lin. Approx. Eq. $y=1$ is a stable equilibrium, but not asymptotically stable.

What about the Nonlinear Diff $\frac{dy}{dt} = (y-1)^3$ near $y \approx 1$?

sign of $f(y)$



$y=1$ is unstable.

The two local dynamics near $y \approx 1$ are different!