Nonlinear System of DDE's
\[ \frac{dx}{dt} = f(x) \]
with equilibrium \( x = \bar{a} \)

Lin. Approx. System near \( x \approx \bar{a} \)
\[ \frac{dx}{dt} = \begin{bmatrix} \text{Jacobian} \end{bmatrix}_{\text{evaluated at } x = \bar{a}}(x - \bar{a}) \]

- The two local dynamics near \( x \approx \bar{a} \) are equivalent, provided that \( \begin{bmatrix} \text{Jacobian} \end{bmatrix} \) has no neutral eigenvalues (i.e. All eigenvalues of \( \begin{bmatrix} \text{Jacobian} \end{bmatrix} \) have \( \text{Re} \lambda \neq 0 \))

- In case \( \begin{bmatrix} \text{Jacobian} \end{bmatrix} \) has neutral eigenvalues, (i.e. \( \text{Re} \lambda = 0 \) for some \( \lambda \))
  the two local dynamics near \( x \approx \bar{a} \)
  may be equivalent / may be non-equivalent.
  - In most cases with neutral eigenvalues, they are Non-Equivalent
  - Need to use other methods to determine
Nonlinear Diff Eq.
\[
\frac{dy}{dt} = f(y)
\]
with equilibrium \(y = b\)

Lin. Approx. Diff Eq Near \(y \approx b\)
\[
\frac{dy}{dt} = f'(y-b)
\]
where \(f'(y)\) = \(\left\frac{df}{dy}\right\) evaluated at \(y = b\)

- The two local dynamics near \(y \approx b\) are equivalent, provided that \(f'(y) \neq 0\).

- In case \(f'(y) = 0\), the two local dynamics near \(y \approx b\) may be equivalent / may be non-equivalent.
- In most cases of \(f'(y) = 0\), they are non-equivalent.
- Need to use other methods to determine
Example  Nonlinear Diff Eq  \( \frac{dy}{dt} = -y^3 + y^2 + 2y \).  \( f(y) = -y^3 + y^2 + 2y \)

• Equilibrium: \( -y^3 + y^2 + 2y = 0 \), \( -y(y^2 - y - 2) = -y(y - 2)(y + 1) \)
  \( y = 0 \), \( y = 2 \), \( y = -1 \).

• Near \( y \approx 0 \): \( \frac{df}{dy} = -3y^2 + 2y + 2 \), \( \frac{df}{dy} \bigg|_{y=0} = 2 \).

Lin. Approx. Diff Eq. Near \( y \approx 0 \):  \( \frac{dy}{dt} = 2y \)
In this Lin. Approx. Eq, \( y = 0 \) is repulsive & unstable
Hence, in the nonlinear diff eq \( \frac{dy}{dt} = f(y) \),
\( y = 0 \) is repulsive & unstable

• Near \( y \approx 2 \): \( \frac{df}{dy} \bigg|_{y=2} = -6 \)
Lin. Approx. Diff Eq. Near \( y \approx 2 \):  \( \frac{dy}{dt} = -6(y - 2) \)
In this Lin. Approx Eq, \( y = 2 \) is attractive & asymp. stable
Hence, in the nonlin. diff eq \( \frac{dy}{dt} = f(y) \),
\( y = 2 \) is attractive & asymp. stable.
Near \( y \approx -1 \):

\[
\left. \frac{df}{dy} \right|_{y=-1} = \left. (-3y^2 + 2y + 2) \right|_{y=-1} = -3
\]

**Lin. Approx. Diff Eq. Near \( y \approx -1 \):**

\[
\frac{dy}{dt} = -3(y + 1)
\]

In this Lin. Approx Eq., \( y = -1 \) is attractive & asymp. stable.

In the Nonlin. Diff Eq

\[
\frac{dy}{dt} = -y^3 + y^2 + 2y
\]

\( y = -1 \) is attractive & asymp. stable.

- We can use the sign-changing method to verify the above:

  sign of \( f(y) = -y^3 + y^2 + 2y \):

  ![Sign diagram]

  \(-1\) \(0\) \(2\)
Example Nonlinear Diff Eq \( \frac{dy}{dt} = (y-1)^3 \) \( f(y) = (y-1)^3 \).

Equilibria: \((y-1)^3 = 0 \Rightarrow y = 1\).

Near \(y \approx 1\): \( \frac{df}{dy} = 3(y-1)^2 \), \( \frac{df}{dy} |_{y=1} = 0 \).

Lin. Approx. Diff Eq. Near \(y \approx 1\): \( \frac{dy}{dt} = 0(y-1) \), i.e. \( \frac{dy}{dt} = 0 \).

In this Lin. Approx. Eq. \( y = 1 \) is a stable equilibrium, but not asymptotically stable.

What about the Nonlinear Diff \( \frac{dy}{dt} = (y-1)^3 \) near \(y \approx 1\)?

**sign of \(f(y)\)**

\[\begin{array}{c|c}
\text{---} & +++ \\
1 & \rightarrow y
\end{array}\]

\(y = 1\) is unstable.

The two local dynamics near \(y \approx 1\) are different.